

Spring 1-1-2011

Spectral Properties of Beta Hermite and Beta Laguerre Ensembles as Beta to Infinity

Michael Bailey Noyes

University of Colorado at Boulder, michael.noyes@colorado.edu

Follow this and additional works at: http://scholar.colorado.edu/math_gradetds



Part of the [Mathematics Commons](#)

Recommended Citation

Noyes, Michael Bailey, "Spectral Properties of Beta Hermite and Beta Laguerre Ensembles as Beta to Infinity" (2011). *Mathematics Graduate Theses & Dissertations*. Paper 7.

This Dissertation is brought to you for free and open access by Mathematics at CU Scholar. It has been accepted for inclusion in Mathematics Graduate Theses & Dissertations by an authorized administrator of CU Scholar. For more information, please contact cuscholaradmin@colorado.edu.

**Spectral Properties of the General β -Hermite and
 β -Laguerre Ensembles in the Limit $\beta \rightarrow \infty$.**

by

Michael Noyes

B.A., New York University, 2001

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

2011

This thesis entitled:
Spectral Properties of the General β -Hermite and β -Laguerre Ensembles in the Limit $\beta \rightarrow \infty$.
written by Michael Noyes
has been approved for the Department of Mathematics

Brian Rider

Prof. Janos Engländer

Date _____

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Noyes, Michael (Ph.D., Mathematics)

Spectral Properties of the General β -Hermite and β -Laguerre Ensembles in the Limit $\beta \rightarrow \infty$.

Thesis directed by Prof. Brian Rider

In 2002, Dumitriu and Edelman introduced three ensembles of tridiagonal random matrix models for a general parameter $\beta > 0$. These ensembles generalized the classical ensembles corresponding to $\beta = 1, 2$, or 4 . The generalization of the behavior of the spectrum for two of these models, the β -Hermite and β -Laguerre, in the regime of the largest or smallest eigenvalue, was proved by Ramírez, Rider, and Virág in 2007.

This thesis describes the behavior of the spectrum of these two ensembles as $\beta \rightarrow \infty$. It is found that the eigenvalues become deterministic, fixing themselves at the roots of the Hermite or Laguerre orthogonal polynomials. When β is large, but not infinite, the eigenvalues have first order Gaussian fluctuations around these roots.

Laws of Large Numbers, Central Limit Theorems and the covariance structure for these eigenvalues are derived. Connections between the work of Dumitriu and Edelman and Ramírez, Rider, and Virág are examined. Directions for future research and open problems are also discussed.

Dedication

To my wife, Adrienne, and our children, Aedan and Fiona.

Acknowledgements

The Department of Mathematics here at CU has been an amazing home for these past five years. Thank you to everybody who made this such a memorable experience. I would like to give special thanks to all the members, past and present, of the CU Probability Group: Prof. Janos Engländer, Prof. Sergei Kuznetsov, Prof. Brian Rider, Prof. Chris Sinclair, Ben Katz-Moses, and William Stanton.

Professor Sergei Kuznetsov taught the classes where I first encountered Probability and Stochastic Processes. He has been a constant presence in my mathematical journey over the past 4 years. It has been a great pleasure and an honor to learn from such a distinguished Probabilist, and I would like to give him my deepest thanks.

Professor Brian Rider has made the research phase of my Graduate studies the most exciting and enjoyable intellectual experience of my life. I cannot adequately thank him for his generosity of time, ideas, and inspiration. I am truly indebted to him for not only teaching me how to do Mathematics, but also for showing me how to be a Mathematician.

Finally, I would like to thank my family. My parents support, love, and, lately, babysitting has been invaluable. I can't thank enough my parents-in-law for all of their help and love. And, of course, to my wife Adrienne and our children, Aedan and Fiona: You have kept my heart as active as my mind. I love you and I thank you for all the beauty and joy you've brought to my life.

Contents

Chapter	
1	Introduction 1
2	Background Material 4
2.1	Basic Results for the Gaussian Ensemble 5
2.2	Basic Results for the Laguerre Ensemble 8
2.3	Tridiagonal Models of Dumitriu and Edelman. 11
2.4	Edelman and Sutton conjectures 13
2.5	Variational argument for the soft edge. 17
2.6	Integral operator for the hard edge. 18
2.7	Dumitriu and Edelman $\beta \rightarrow \infty$ paper. 19
3	Large β : Hermite Ensemble 24
3.1	Introduction 24
3.2	β -Hermite as $\beta \rightarrow \infty$ 24
3.3	Commuting the limits 36
3.4	Random Differential Operator 40
4	Large β : Laguerre Ensemble 44
4.1	Introduction 44
4.2	(β, a) -Laguerre with n fixed and $\beta \rightarrow \infty$ 44

4.3	(β, a) -Laguerre for $n \rightarrow \infty$ and $\beta \rightarrow \infty$	49
4.4	Commuting the limits.	54
4.5	Transition from Hard Edge to Soft as $a \rightarrow \infty$	58
5	Future Work and Open Problems	62
5.1	$\beta \rightarrow 0$	62
5.2	Resolvent	65
	Bibliography	76

Chapter 1

Introduction

Random Matrix Theory (RMT) began in the work of mathematical statisticians in the 1920's on problems involving large sample covariance matrices. The subject did not develop much until the mathematical physicist Eugene Wigner proposed in the 1950's that the eigenvalues of large random hermitian matrices could be used to model the scattering resonances of neutrons off of heavy nuclei. This matched the experimental evidence incredibly well and the subject underwent intense development in the mathematical physics community. It wasn't until the 1990's that the subject started gaining the attention of the broader mathematical community, with one of the first and most important breakthroughs being the discovery of a new class of probability distributions named after their discoverers, Craig Tracy and Harold Widom. These Tracy-Widom distributions have been shown to describe the limiting behavior for a vast array of diverse objects, and much current research is devoted to explicating as many of their properties as possible.

In the physical context, the ensembles of random matrices that were studied were parameterized by one of the three values $\beta = 1, 2, 4$. These arose from the quantum mechanical properties of the system under consideration. From a mathematical point of view, these three particular values lead to eigenvalue particle systems that are known as Integrable, which means that all of the correlation functions are explicitly computable. From a knowledge of the correlation functions, very fine local limit theorems can be formulated. All of the statistical quantities of interest can be recovered from the correlation functions, so having systems that are integrable is clearly very desirable. Unfortunately, only those models with $\beta = 1, 2, 4$ are, and for many years all work focused

exclusively on these three parameter values.

It was a huge breakthrough, then, when in 2002 Ioana Dumitriu and Alan Edelman introduced ensembles of symmetric tridiagonal matrices whose joint eigenvalue densities were generalizations to all $\beta > 0$, of the well-known and studied densities for $\beta = 1, 2, 4$. This allowed the obstacles to development due to the lack of integrability to be bypassed, since the focus could now be on the matrices themselves. Many of the “classical” results have been generalized to arbitrary $\beta > 0$. A natural question presented itself: what happens in the limit as $\beta \rightarrow \infty$? This thesis provides some answers to that question in two specific ensembles, the β -Hermite and the (β, a) -Laguerre.

Here is an outline of the rest of the thesis.

Chapter 2 contains the relevant preliminary mathematical results from RMT that this thesis builds on. Unless the proof of the stated theorems are explicitly needed later on, they are merely referenced. After a brief introduction to the key concepts and theorems concerning the Hermite and Laguerre ensembles, the work of Dumitriu and Edelman on the Tridiagonal models is considered. The Stochastic Differential Operator Conjectures of Edelman and Sutton are derived, and the work of Ramírez, Rider, and Virág in proving the Edelman and Sutton conjectures is described. Finally, the work of Dumitriu and Edelman on the limit as $\beta \rightarrow \infty$ for finite $n \times n$ matrices from the β -Hermite and (β, a) -Laguerre ensembles is presented. This chapter also serves to fix notation.

Chapter 3 contains results for the β -Hermite ensembles after the parameter β and then, n , the size of the matrix, goes to infinity. These results include a Law of Large Numbers and Central Limit Theorems for the convergence of the eigenvalues to the roots of Hermite polynomials. The vector of eigenvalues for a fixed rank matrix converges to a multivariate Gaussian random variable with explicit covariance matrix, as $\beta \rightarrow \infty$. As the size of the matrix is taken to infinity, the vector of eigenvalues converges to an infinite Gaussian process. The (infinite) covariance matrix is calculated, and the asymptotic decay rates of the variance and covariance of the eigenvalues are derived. Finally, it is shown that one can take the size of the matrix to infinity first and then let $\beta \rightarrow \infty$, that is, that the β and n limits commute.

Chapter 4 follows the same blueprint as chapter 3, except it examines the Laguerre Ensembles.

These ensembles depend on two parameters, a and β . The main results again are a Law of Large Numbers and Central Limit Theorem for the eigenvalues. Covariance matrices in both the finite dimensional and infinite dimensional settings are obtained. The question of commuting the $\beta - n$ limits is also explored, though, it proves to be much harder to analyze than in the β -Hermite case. Finally, the transition from the (β, a) -Laguerre covariance matrix to the β -Hermite covariance matrix as $a \rightarrow \infty$ is presented.

In Chapter 5, two related topics are examined. The first is the limiting spectral behavior for these ensembles as $\beta \rightarrow 0$. This situation is the exact opposite of the $\beta \rightarrow \infty$ case already considered in that now the randomness is becoming infinite. The conjectured behavior is that the repulsion (and therefore structure) built into the eigenvalues is washed-out by the randomness, and that the eigenvalues become a Poisson point process. The second section explores the possibility of constructing a resolvent operator which would generalize the integral operator given in [RR09]. This is closely connected to properties of orthogonal polynomials, which are investigated in some detail. There are a few original results and conjectures in this chapter, but they do not form a cohesive whole the way that the $\beta \rightarrow \infty$ results do.

The heart of this thesis is found in chapters 3 and 4. If a given result is not explicitly referenced it is original.

Chapter 2

Background Material

Random Matrix Theory (RMT) is concerned with the statistical properties of certain families, called ensembles, of random matrices. These are matrices, H , whose entries are random variables. For mathematical and physical reason, the matrices are assumed to have some sort of symmetry, usually that H is symmetric, $H = H^T$, if the entries are real random variables, or self-adjoint, $H = H^*$, if the entries are complex random variables.

The spectral theory of random matrices studies the distribution of the eigenvalues as the size of the matrix goes to infinity. In the *global regime*, one is interested in the empirical spectral measure

$$\mu_{H,n}(A) = \frac{1}{n} \# \{ \text{eigenvalues of } H \text{ in } A \}, \quad A \subset \mathbb{R}. \quad (2.1)$$

As the size of the matrix goes to infinity, $\mu_{H,n} \Rightarrow \mu_H$, where μ_H is a deterministic measure called the limiting spectral measure.

In the *local regime*, the objects of interest are the spacings between eigenvalues, and, more generally, the joint distribution of eigenvalues in an interval of order $1/n$. In this regime, a distinction is made between *bulk* statistics, which pertain to intervals inside the support of the limiting spectral measure μ_H , and *edge* statistics, which pertain to intervals near the boundary of the boundary of the limiting spectral measure.

The two ensembles considered in this thesis are the Gaussian ensembles, also called the β -Hermite ensembles, and the Wishart ensembles, also called the (β, a) -Laguerre ensemble. The

following will provide a brief introduction to these ensembles, the main theorems concerning the global and local behavior of the eigenvalues, and the more recent results that the research contained in this thesis is based on.

2.1 Basic Results for the Gaussian Ensemble

Let $\{x_{i,j}, y_{i,j}\}_{i,j=1}^{\infty}$ be i.i.d. standard Gaussian random variables. Consider the two collections of $n \times n$ matrices, denoted $\mathcal{H}_n^{(\beta)}$ with $\beta = 1, 2$, of all matrices of the following forms:

$$X^{(1)} = \begin{pmatrix} \sqrt{2}x_{1,1} & x_{1,2} & x_{1,3} & \dots \\ x_{1,2} & \sqrt{2}x_{2,2} & x_{2,3} & \dots \\ \vdots & \ddots & \ddots & \dots \\ \dots & x_{n-2,n} & x_{n-1,n} & \sqrt{2}x_{n,n} \end{pmatrix} \quad (2.2)$$

and

$$X^{(2)} = \begin{pmatrix} x_{1,1} & \frac{x_{1,2}+iy_{1,2}}{\sqrt{2}} & \frac{x_{1,3}+iy_{1,3}}{\sqrt{2}} & \dots \\ \frac{x_{1,2}-iy_{1,2}}{\sqrt{2}} & x_{2,2} & \frac{x_{2,3}+iy_{2,3}}{\sqrt{2}} & \dots \\ \vdots & \ddots & \ddots & \dots \\ \dots & \frac{x_{n-2,n}-iy_{n-2,n}}{\sqrt{2}} & \frac{x_{n-1,n}-iy_{n-1,n}}{\sqrt{2}} & x_{n,n} \end{pmatrix} \quad (2.3)$$

Let the law of these matrices be denoted by $P_n^{(\beta)}$. A random matrix $X^{(\beta)} \in \mathcal{H}_n^{(\beta)}$ with law $P_n^{(\beta)}$ is said to belong to the Gaussian Orthogonal Ensemble (GOE) when $\beta = 1$ or the Gaussian Unitary Ensemble (GUE) when $\beta = 2$. The fundamental result about the global behavior of the spectrum for GOE and GUE is the following.

Theorem 2.1.1. (Wigner's Semicircle Law [Wig55]) *Let $\lambda_1^n \leq \lambda_2^n \leq \dots \leq \lambda_n^n$ denote the ordered eigenvalues of $\frac{1}{\sqrt{n}}X_n$, for X_n a GOE or GUE matrix. Then, with the convergence being weakly almost surely,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i^n} = \sigma(x) \quad (2.4)$$

where $\sigma(x)$ is the probability distribution on \mathbb{R} with density

$$\sigma(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{1}_{|x| \leq 2}.$$

This is called the semicircle distribution.

Remark 1. Wigner's theorem holds in much more generality than the version presented above. In fact, other than the symmetry condition, the only assumptions needed are that the random variables filling the matrix be i.i.d., mean zero, the off-diagonal entries have variance 1, and all have finite moments. If these conditions are met, then Wigner's theorem applies and the empirical measure converges to the semicircle distribution. This is an example of what is known as universality.

To investigate the local behavior of the eigenvalues in GOE and GUE, the starting point is the joint distribution of the eigenvalues. For any vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$, let

$$\Delta(\mathbf{x}) = \det(\{x_i^{j-1}\}_{i,j=1}^n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

This is called the Vandermonde determinant. The joint distribution of eigenvalues is absolutely continuous with respect to Lebesgue measure, with density given by

$$f_\beta(\lambda_1, \dots, \lambda_n) = Z_{\beta,n} |\Delta(\lambda)|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^n \lambda_i^2} \quad \text{for } \beta = 1, 2. \quad (2.5)$$

$Z_{\beta,n}$ is an explicitly known normalization constant. While this density function is well-defined for all $\beta > 0$, when $\beta = 1, 2, 4$, it has many special properties. Perhaps the most important of these concerns the correlation functions. Define the k -point correlation function, $R_k(\lambda_1, \dots, \lambda_k)$, as

$$R_k(\lambda_1, \dots, \lambda_k) = \frac{n!}{(n-k)!} \int f_\beta(\lambda_1, \dots, \lambda_n) d\lambda_{k+1} \dots d\lambda_n. \quad (2.6)$$

These functions have the following interpretation:

$$\int_B R_1(\lambda_1) d\lambda_1 = \text{expectation of the number of eigenvalues in } B \quad (2.7)$$

$$\int_{B \times B} R_2(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = \text{expectation of the number of pairs of eigenvalues in } B, \quad (2.8)$$

and so on. Then, in theory, knowledge of all n -point correlation functions allows any statistical quantity of interest concerning the eigenvalues to be calculated.

Using the theory of orthogonal polynomials and the representation of $\Delta(\mathbf{x})$ as a determinant, it can be shown that

$$f_\beta(\lambda_1, \dots, \lambda_n) = \det (K_n(\lambda_i, \lambda_j))_{1 \leq i, j \leq n}, \quad (2.9)$$

where

$$K_n(x, y) = \sum_{j=0}^{n-1} \phi_j(x) \phi_j(y), \quad (2.10)$$

is called the kernel, and where $\phi_i(x)$ is an arbitrary polynomial. The proper choice for the family $\phi_j(x)$ are the normalized Hermite polynomials. This is because of the presence of the weight, $e^{-x^2/2}$ (in the $\beta = 2$ case), for the Hermite polynomials in (2.5). Using the properties of orthogonal polynomials and some formulas relating integrals of determinant with determinants of integrals, it can be shown that the kernels have the following ‘reproducing property,’

$$\int K_n(x, y) K_n(y, z) dy = K_n(x, z). \quad (2.11)$$

This, combined with (2.6), gives the explicit formula

$$R_k(\lambda_1, \dots, \lambda_k) = \det (K_n(\lambda_i, \lambda_j))_{1 \leq i, j \leq k} \quad (2.12)$$

for the n -point correlation functions. With this formula in hand, the statistical properties of the spectrum now allow for fine analysis, leading to some remarkable theorems, the most important being the Tracy-Widom distribution for the largest eigenvalue, presented here for GUE.

Theorem 2.1.2. (Tracy-Widom Distribution for GUE [TW94a]) *Let λ_n^n denote the largest eigenvalue of a matrix from the Gaussian Unitary Ensemble. Then*

$$\lim_{n \rightarrow \infty} P \left(n^{2/3} \left(\frac{\lambda_n^n}{\sqrt{n}} - 2 \right) < t \right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_t^{\infty} \cdots \int_t^{\infty} \det (K_{Airy}(x_i, x_j))_{i, j=1, \dots, k} dx_1 \cdots dx_k \quad (2.13)$$

$$= \exp \left(- \int_t^\infty (x-t)q(x)^2 dx \right), \quad (2.14)$$

where K_{Airy} is the Airy Kernel defined by

$$K_{\text{Airy}}(x, y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x - y}, \quad (2.15)$$

and where q is a solution to the Painlevé II equation.

Remark 2. *There are corresponding Tracy-Widom distributions for the GOE and GSE, given in terms of integrals of solutions to other Painleve equations. There are also general β Tracy-Widom laws, although there are not explicit formulas when $\beta \neq 1, 2$, or 4 . These laws have emerged as important new families of probability distribution due to their appearance in a wide-range of situations.*

Remark 3. *When all the n -point correlation functions are computable, the system is called Integrable, as mentioned in the Introduction. There is a vast body of literature and many well-developed techniques for analyzing such systems. The prominence of the values $\beta = 1, 2, 4$ in the classical results of RMT is that all of these systems are Integrable, and therefore amenable to analysis. For values of β outside of $1, 2, 4$, the systems are not Integrable, and this provided the primary obstacle to the development of a general β theory.*

Since the Gaussian Ensembles have global behavior that is governed by the Semicircle Law, the behavior of the largest and smallest eigenvalue is symmetric, which means that the Tracy-Widom distribution describes the behavior of the smallest eigenvalue as well. For reasons that will be better understood in the context of the next section, these are called soft edges.

2.2 Basic Results for the Laguerre Ensemble

Let $M = M(n)$ be a sequence of positive integers such that

$$\lim_{n \rightarrow \infty} \frac{M(n)}{n} = a \in [1, \infty) \quad (2.16)$$

Let A_n be an $n \times M(n)$ matrix with i.i.d. entries of mean zero, variance $1/n$ and satisfying certain moment conditions. If the entries are real, then $\beta = 1$, while if they are complex, $\beta=2$. Then the collection of $n \times n$ matrices $W_n = A_n A_n^T$ is called the Wishart, or Laguerre, ensemble. Since W_n is a positive matrix, all the (real) eigenvalues are greater than zero. The global spectral behavior is given by the following theorem.

Theorem 2.2.1. (Marchenko-Pastur Distribution [MP67]) *Let $0 < \lambda_1^n < \lambda_2^n < \dots \lambda_n^n$ denote the ordered eigenvalues of a Wishart matrix, W_n . Then, with the convergence being weakly, in probability*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^N \delta_{\lambda_i^n} = F_a \quad (2.17)$$

where F_a is a distribution function with density

$$f_a(x) = \frac{\sqrt{(x - b_-)(b_+ - x)}}{2\pi x} \mathbb{1}_{[b_-, b_+]}(x) \quad (2.18)$$

with $b_- \equiv (1 - \sqrt{a})^2$ and $b_+ \equiv (1 + \sqrt{a})^2$.

Remark 4. *This $f_a(x)$ is the Marchenko-Pastur distribution. The dependence on the parameter a is clearly visible. This is a measure of the difference in size of the rectangular matrix A_n . When $a = 1$, the matrices A_n are asymptotically square, and F_1 is the image of the semicircle distribution under the transformation $x \rightarrow x^2$. This law is another example of universality since there are no restrictions, other than certain growth conditions, put on the distribution of the matrix entries.*

The joint distribution of eigenvalues for the (β, a) -Laguerre ensemble is

$$f_{\beta,a}(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_{\beta,a,n}} |\Delta(\lambda)|^\beta \prod_{k=0}^{n-1} \lambda_k^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2} \sum_{i=1}^n \lambda_i}, \quad (2.19)$$

where $Z_{\beta,a,n}$ is an explicitly known normalization constant that depends on both β and a . Due to the positivity of these matrices, the behavior of the smallest eigenvalue is very different from the behavior of the largest. The smallest eigenvalue is constrained to be positive, and so it feels the hard edge at the origin. The largest eigenvalue has no similar constraint, so it is said to be at the

soft edge, just as in the Gaussian Ensembles, and it's behavior is governed by the Tracy-Widom distribution. The basic result describing the behavior of the smallest eigenvalue, for $\beta = 2$, is given by the following theorem, also due to Tracy and Widom:

Theorem 2.2.2. (Tracy-Widom Hard Edge Distribution [TW94b]) *Denote by $\lambda_{0,a}$ the smallest (positive) eigenvalue of (β, a) -Laguerre ensemble, for $\beta = 2$. Then,*

$$P(n\lambda_{0,a} > t) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_0^t \cdots \int_0^t \det(K_{Bessel}^a(x_i, x_j))_{i,j=1,\dots,k} dx_1 \cdots dx_k \quad (2.20)$$

$$= \exp\left(-\frac{1}{4} \int_0^t \log\left(\frac{t}{x}\right) p(x)^2 dx\right) \quad (2.21)$$

where K_{Bessel}^a is the Bessel Kernel defined by

$$K_{Bessel}^a = \frac{\sqrt{y}J_a(\sqrt{x})J'_a(\sqrt{y}) - \sqrt{x}J_a(\sqrt{y})J'_a(\sqrt{x})}{x - y}, \quad (2.22)$$

and $p(x)$ is a solution to the Painlevé V equation.

Remark 5. *For $\beta = 1, 2, 4$ these systems are Integrable in the sense described above. The derivations of the correlation functions will not be presented, but it again hinges upon orthogonal polynomials, in this instance the Laguerre polynomials. This is due to the presence of the Laguerre weight, $x^a e^{-x}$ (in the $\beta = 2$ case), in (2.19). This explains the terminology (β, a) -Laguerre ensemble. When $\beta \neq 1, 2, 4$, the systems are not Integrable.*

Finally, if the parameter a is taken to infinity after $n \rightarrow \infty$, the hard edge “pulls” away from the origin, and one sees the semicircle distribution. Explicitly, if $\lambda_{0,2a}$ denotes the smallest eigenvalue of the $(\beta, 2a)$ -Laguerre ensemble, then, for any $\beta > 0$,

Theorem 2.2.3. (Theorem 3 of [RR09]) *With $\lambda_{0,2a}$ as above and TW_β the Tracy-Widom distribution with parameter β ,*

$$\frac{a^2 - \lambda_{0,2a}}{a^{4/3}} \Rightarrow TW_\beta, \quad (2.23)$$

as $a \rightarrow \infty$.

This is saying that as $a \rightarrow \infty$, the distribution of the smallest eigenvalue converges to the Tracy-Widom distribution.

2.3 Tridiagonal Models of Dumitriu and Edelman.

Define the β -Hermite ensemble, for arbitrary $\beta > 0$, as the ensemble of all symmetric tridiagonal matrices

$$A_\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} N(0, 2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0, 2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0, 2) & \chi_\beta \\ & & & \chi_\beta & N(0, 2) \end{pmatrix} \quad (2.24)$$

where χ_i is a chi random variable (not chi-square), and $N(0, 2)$ denotes a Gaussian with mean zero and variance 2. Up to symmetry, all random variables are independent. The importance of this ensemble is contained in the following theorem.

Theorem 2.3.1. (Theorem 2.12 of [DE02]) *With the notation as above, for any $\beta > 0$, A_β has joint eigenvalue density*

$$f_\beta^H(\lambda_1, \lambda_2, \dots, \lambda_n) = \frac{1}{Z_{\beta,n}} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta e^{-\sum_{i=1}^n \lambda_i^2/2}. \quad (2.25)$$

When $\beta = 1, 2$ this is exactly the G(O/U)E eigenvalue density function seen in (2.5).

The basic idea behind the proof is easiest to see in the particular case $\beta = 1$. Let X_n be defined as in (2.2). Set $\eta_n = X_n(1, 1)/\sqrt{2}$. Let $X_n^{(1,1)}$ be the matrix obtained from X_n by removing the first row and first column, and let $Z_{n-1}^T = (X_n(1, 2), \dots, X_n(1, n))$. Then $X_n^{(1,1)}$ is independent of Z_{n-1} . Chose \tilde{H}_n to be an orthogonal $(n-1) \times (n-1)$ matrix such that $\tilde{H}_n Z_{n-1} = (||Z_{n-1}||_2, 0, \dots, 0)$,

and let $Y_{n-1} = ||Z_{n-1}||_2$. Let

$$H_n = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{H}_n \end{pmatrix}. \quad (2.26)$$

Then, the distribution of eigenvalues of $H_n X_n H_n^T$ is the same as of X_n , but

$$H_n X_n H_n^T = \begin{pmatrix} \sqrt{2}\eta_n & Y_{n-1} & \mathbf{0}_{n-2} \\ Y_{n-1} & & \\ & X_{n-1} & \\ \mathbf{0}_{n-2} & & \end{pmatrix}, \quad (2.27)$$

where X_{n-1} is distributed according to GOE and is independent of η_n and Y_{n-1} . Iterating this construction $n - 1$ times leads to the desired tridiagonal form.

Remark 6. *One can always chose \tilde{H}_n to be the Householder reflector $\tilde{H}_n = Id - 2uu^T / ||u||_2^2$, where $u = Z_{n-1} - ||Z_{n-1}||_2(1, 0, \dots, 0)$. This was originally done, for $\beta = 1, 2$, in [Tro84].*

The (β, a) -Laguerre ensemble is slightly more complicated. There are now two parameters, $\beta > 0$ and $a > -1$. The (β, a) -Laguerre ensemble is defined as all matrices of the form $L_{\beta, a} = B_{\beta, a} B_{\beta, a}^T$, where

$$B_{\beta, a} = \begin{pmatrix} \chi_{(a+n)\beta} & & & \\ \chi_{(n-1)\beta} & \chi_{(a+n-1)\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{(a+1)\beta} \end{pmatrix} \quad (2.28)$$

Again, the eigenvalues of the matrix $L_{\beta, a}$ have joint density function which generalizes the classical cases:

Theorem 2.3.2. (Theorem 3.4 of [DE02]) *For any $\beta > 0$ and $a > -1$, $B_{\beta, a}$ has joint eigenvalue density given by*

$$f_{\beta, a}^L(\lambda_1, \lambda_2, \dots, \lambda_n) = \frac{1}{Z_{\beta, a, n}} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^n \lambda_i^{\frac{\beta}{2}(a+1)-1} e^{-\frac{\beta}{2} \sum_{i=1}^n \lambda_i} \quad (2.29)$$

It is worth remarking that the weights for the Hermite and Laguerre orthogonal polynomials are explicit in these two expressions. This allows the theory of orthogonal polynomials to be brought to bear, and also gives rise to the alternate names of these general- β ensembles.

2.4 Edelman and Sutton conjectures

In [ES07], Alan Edelman and Brian Sutton made the following conjectures connecting the eigenvalues of the β -Hermite and (β, a) -Laguerre ensembles to the eigenvalues and singular values of the following random differential operators.

Conjecture 1. *As $n \rightarrow \infty$, the rescaled and centered matrix $n^{\frac{1}{6}}(2\sqrt{n} \text{Id}_n - A_\beta)$ converges to the following stochastic differential operator*

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}}b'(x). \quad (2.30)$$

This was dubbed the Stochastic Airy Operator since the $\beta = \infty$ limit corresponds to the deterministic Airy differential equation.

Conjecture 2. *Let σ_k denote the k -th smallest singular value of the bidiagonal matrix $B_{\beta,a}$. As $n \rightarrow \infty$, the family of rescaled singular values $\{\sqrt{n}\sigma_k\}$ converges in law to the singular values of the following random differential operator*

$$\mathcal{L}_{\beta,a} = -\sqrt{x}\frac{d}{dx} + \frac{a}{2\sqrt{x}} + \frac{1}{\sqrt{\beta}}b'(x). \quad (2.31)$$

This was tagged the Stochastic Bessel Operator on account of the $\beta = \infty$ limit having singular values at the roots of the Bessel function of the first kind $J_a(x)$.

The main reason why these are formal differential operators is the presence of the white noise term. Recall that on a formal level, white noise is defined as the derivative of Brownian motion, $b'(t)$. In light of the non-differentiability of Brownian motion, this leads to immediate problems. To make sense of this derivative, one has to consider it as a distribution (generalized function, not in the probabilistic sense). Then, since Brownian motion is continuous, $b'(t)$ is defined as the random

distribution which maps $\varphi \in C_0^\infty$

$$\varphi \rightarrow \int \varphi(t)b'(t)dt = \int \varphi(t)db(t). \quad (2.32)$$

The final integral is a well-defined Itô integral, and it is distributed as $N(0, \int \varphi^2)$. This distribution definition suggests that the appropriate way to handle the white noise terms is through integration, and specifically, integration by parts. This will be discussed in more detail below.

For the purposes of what follows, only the details of the Stochastic Airy Operator will be considered. The derivation of the Stochastic Bessel Operator is similar, and the full details can be found in the original paper of Edelman and Sutton.

Start with A_β , for a fixed n . By Wigner's Semicircle Theorem 2.1.1, on average the largest eigenvalue of this matrix is close to $2\sqrt{n}$. Denote this eigenvalue by λ_n . Then in order to discover the limiting behavior of λ_n , the appropriate rescaling and centering turn out to be

$$n^{1/6}(2\sqrt{n}\text{Id}_n - A_\beta). \quad (2.33)$$

Since the entries of the matrix decrease along the off-diagonal, and since it is the largest eigenvalue that is being considered, one would expect most of the contribution to come from the upper-left corner of the matrix, where the random variables are the largest. When $n \gg 1$ is an integer, it follows from the Central Limit Theorem that $\chi_n \sim \sqrt{n} + G$ with G a Gaussian random variable with mean zero and variance $1/2$. It follows that in the upper-left corner, where $n \gg k$, the off-diagonal entries are of the form

$$\frac{1}{\sqrt{\beta}}\chi_{(n-k)\beta} \sim \sqrt{n-k} + \frac{1}{\sqrt{2\beta}}G_k \sim \sqrt{n} - \frac{k}{2\sqrt{n}} + \frac{1}{\sqrt{2\beta}}G_k \quad (2.34)$$

where the family $\{G_i\}_{i=1}^{n-1}$ are i.i.d. random variables distributed according to $N(0, 1/2)$. Letting $\{N_i\}_{i=1}^n$ denote i.i.d. standard Gaussian random variables, the rescaled and centered matrix can

be written as

$$n^{1/6}(2\sqrt{n}\text{Id}_n - A_\beta) \sim n^{2/3} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix} + \frac{1}{2n^{1/3}} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 2 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & \ddots & n-1 \\ & & & n-1 & 0 \end{pmatrix} \quad (2.35)$$

$$+ \frac{n^{1/6}}{\sqrt{2\beta}} \begin{pmatrix} 2N_1 & G_1 & & & \\ G_1 & 2N_2 & G_2 & & \\ & G_2 & \ddots & \ddots & \\ & & \ddots & \ddots & G_{n-1} \\ & & & G_{n-1} & 2N_n \end{pmatrix} \quad (2.36)$$

The idea now is to consider these three matrices as finite difference approximations to operators. In order to do this, assume that f is a twice differentiable function. On the finite mesh $\{h, 2h, 3h, \dots\}$, approximate f by $f_k \equiv f(kh)$. As the mesh size becomes smaller, f_i becomes a better and better approximation.

Now consider the action of each of the three matrices above on the discretized f_i . The first,

$$n^{2/3} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix} \quad (2.37)$$

is the negative of the discretized second derivative operator with $h = n^{-1/3}$. As $n \rightarrow \infty$, this matrix converges to $-\frac{d^2}{dx^2}$. This leads to the first term in the Edelman-Sutton conjecture.

The second matrix, after moving the $n^{-1/3}$ inside, is

$$\frac{1}{2} \begin{pmatrix} 0 & \frac{1}{n^{1/3}} & & & \\ \frac{1}{n^{1/3}} & 0 & \frac{2}{n^{1/3}} & & \\ & \frac{2}{n^{1/3}} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{n-1}{n^{1/3}} \\ & & & \frac{n-1}{n^{1/3}} & 0 \end{pmatrix}. \quad (2.38)$$

When the mesh size is small, f_k and f_{k+1} are close together and the terms can be moved from the off-diagonal to the main diagonal with negligible error. After this change, the second term becomes

$$\begin{pmatrix} \frac{1}{n^{1/3}} & 0 & & \\ 0 & \frac{2}{n^{1/3}} & 0 & \\ & 0 & \frac{3}{n^{1/3}} & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad (2.39)$$

On the mesh of size $h = n^{-1/3}$, this matrix is the discretized version of multiplication by x . This leads to the second term in the conjecture.

Finally, for the random term, consider the third matrix. Again, the off-diagonal terms can be moved to the diagonal to give

$$\frac{n^{1/6}}{\sqrt{2\beta}} \begin{pmatrix} 2N_1 & G_1 & & & \\ G_1 & 2N_2 & G_2 & & \\ & G_2 & \ddots & \ddots & \\ & & \ddots & \ddots & G_{n-1} \\ & & & G_{n-1} & 2N_n \end{pmatrix} \sim \frac{2n^{1/6}}{\sqrt{\beta}} \begin{pmatrix} g_1 & 0 & & & \\ 0 & g_2 & 0 & & \\ & 0 & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & 0 & g_n \end{pmatrix} \quad (2.40)$$

where the g_i are i.i.d. standard Gaussians, independent from all of the other Gaussians. The point now is that this matrix is a discretized version of the operator that multiplies f by $2/\sqrt{\beta}n^{1/6}g_i$. Now, $n^{1/6}g_i \sim N(0, n^{1/3}) = N(0, h^{-1})$, again since our mesh size is $h = n^{-1/3}$. Recalling the definitions of Brownian motion, this gives

$$\frac{b(x+h) - b(x)}{h} \sim N(0, h^{-1}) \quad (2.41)$$

and formally, for h small (so n large), the left-hand side of the above is the difference quotient for the derivative of Brownian motion, $b'(x)$. This completes the derivation of the Stochastic Airy Operator.

2.5 Variational argument for the soft edge.

Let $L^* = \{f : f(0) = 0 \text{ and } \int_0^\infty (f')^2 + (1+x)^2 f^2 dx < \infty\}$. Notice that $L^* \subset L^2$. In this space one has the natural norm

$$\|f\|_*^2 = \int_0^\infty (f')^2 + (1+x)f^2 dx.$$

Define $\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}}b'_x$, where b' indicates white noise. This operator acts in the following way

$$\mathcal{H}_\beta f = -f'' + \frac{2}{\sqrt{\beta}}f + xf. \quad (2.42)$$

This is called the Stochastic Airy Operator on account of the $\beta = \infty$ limit corresponding to the Airy differential equation.

An element $(\psi, \lambda) \in L^* \times \mathbb{R}$ is an eigenfunction/eigenvalue pair for \mathcal{H}_β if $\|\psi\|_2 = 1$ and

$$\psi''(x) = \frac{2}{\sqrt{\beta}}\psi(x)b'_x + (x - \lambda)\psi(x)$$

where the above has to be interpreted in an 'integration-by-parts sense';

$$\psi'(x) = \frac{2}{\sqrt{\beta}}\psi(x)b_x + \int_0^x -\frac{2}{\sqrt{\beta}}b_y\psi'(y)dy + \int_0^x (y - \lambda)\psi(y)dy.$$

The Edelman-Sutton Stochastic Airy Operator conjecture is then proven by the following theorem.

Theorem 2.5.1. (Theorem 1.1 in [RRV]). *With probability one, for each $k \geq 0$ the set of eigenvalues of \mathcal{H}_β has a well-defined $(k+1)$ st lowest element Λ_k . Moreover, let $\lambda_1 \geq \lambda_2 \geq \dots$ denote the eigenvalues of β -Hermite ensemble. Then the vector*

$$\left(n^{1/6}(2\sqrt{n} - \lambda_l)\right)_{l=1,\dots,k} \quad (2.43)$$

converges in distribution as $n \rightarrow \infty$ to $(\Lambda_0, \Lambda_1, \dots, \Lambda_{k-1})$.

The limit law of $-\Lambda_0(\beta)$ is called TW_β , since when $\beta = 1, 2, 4$ it leads to the classical Tracy-Widom distribution for the distribution of the largest eigenvalue.

For the variational approach, it is more convenient to introduce a weak definition of eigenfunction/eigenvalue pairs. It takes the following form:

$$\int \phi'' f dx = \int (x - \lambda) \phi f dx + \int \left[\int_0^x b_y f'(y) dy - b_x f(x) \right] \phi' dx.$$

After an integration-by-parts this can be written

$$\int \phi'' f dx = \int (x - \lambda) \phi f - b f' \phi - b f \phi' dx.$$

Let

$$\tilde{\Lambda}_0 := \inf \{ \langle f, \mathcal{H}_\beta f \rangle_* : f \in L^* \text{ and } \|f\|_2 = 1 \}.$$

Then $\tilde{\Lambda}_0 = \Lambda_0$, the smallest eigenvalue of \mathcal{H}_β . Working with the same variational principle, one can also show that $\tilde{\Lambda}_k$ exists and is equal to the k -th smallest eigenvalue of \mathcal{H}_β , Λ_k .

2.6 Integral operator for the hard edge.

In the (β, a) -Laguerre case, the scaled eigenvalues converge with probability 1 to the eigenvalues of a random differential operator, $\mathfrak{G}_{\beta, a}$, given by

$$-\mathfrak{G}_{\beta, a} = \exp[(a+1)x + \frac{2}{\sqrt{\beta}}b(x)] \frac{d}{dx} \left\{ \exp[-ax - \frac{2}{\sqrt{\beta}}b(x)] \frac{d}{dx} \right\}. \quad (2.44)$$

Here $b(x)$ is a Brownian motion and $a > -1$ and $\beta > 0$. This operator generates the diffusion with speed and scale measures

$$m(dx) = e^{-(a+1)x - \frac{2}{\sqrt{\beta}}b(x)} dx \quad (2.45)$$

$$s(dx) = e^{ax + \frac{2}{\sqrt{\beta}}b(x)} dx. \quad (2.46)$$

It is possible to invert $\mathfrak{G}_{\beta,a}$, which gives the integral operator

$$(\mathfrak{G}_{\beta,a}^{-1}\psi)(x) \equiv \int_0^\infty \left(\int_0^{x \wedge y} e^{az + \frac{2}{\sqrt{\beta}}b(z)} dz \right) \psi(y) e^{-(a+1)y - \frac{2}{\sqrt{\beta}}b(y)} dy. \quad (2.47)$$

This is the resolvent operator $(\lambda - \mathfrak{G}_{\beta,a})^{-1}$ at $\lambda = 0$. With this, the Stochastic Bessel conjecture becomes the following theorem:

Theorem 2.6.1. (Theorem 1 of [RR09]) *With probability one, when restricted to the positive half-line with Dirichlet conditions at the origin, $\mathfrak{G}_{\beta,a}$ has discrete spectrum comprised of simple eigenvalues $0 < \Lambda_0(\beta, a) < \Lambda_1(\beta, a) < \dots \uparrow \infty$. Moreover, with now, $0 < \lambda_0 < \lambda_1, \dots < \lambda_n$, the ordered (β, a) -Laguerre eigenvalues,*

$$\{n\lambda_0, n\lambda_1, \dots, n\lambda_k\} \Rightarrow \{\Lambda_0(\beta, a), \Lambda_1(\beta, a), \dots, \Lambda_k(\beta, a)\} \quad (2.48)$$

(jointly in law) for any fixed $k < \infty$ as $n \uparrow \infty$.

2.7 Dumitriu and Edelman $\beta \rightarrow \infty$ paper.

In the paper [DE05], Dumitriu and Edelman consider the spectrum of a fixed $n \times n$ matrix from the β -Hermite or (β, a) -Laguerre as $\beta \rightarrow \infty$. In the thermodynamical context, where β corresponds to inverse temperature, this limit has the interpretation of the system of n particles freezing into place. Mathematically, as $\beta \rightarrow \infty$ the eigenvalues (in this case there are only n), become deterministic, freezing at the roots of the Hermite or Laguerre polynomials, for the β -Hermite and the (β, a) -Laguerre ensembles, respectively. When β is very large but not infinite, the eigenvalues have a Gaussian distribution to the first order around the roots of the corresponding orthogonal polynomials.

Most of the results in [DE05] are based on the following two lemmas:

Lemma 2.7.1. (Theorem 4.4 in [Dem97])

Let A and B be $n \times n$ symmetric matrices, and let $\epsilon > 0$. Assume A has all distinct eigenvalues. Let $M = A + \epsilon B + o(\epsilon)$, where $o(\epsilon)$ denotes a matrix in which every entry goes to 0 faster than ϵ .

Let $\lambda_i(X)$ denote the i th eigenvalue of X , for $1 \leq i \leq n$. Finally, let Q be an eigenvector matrix for A . Then

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\lambda_i(M) - \lambda_i(A)) = Q_i^T B Q_i$$

where Q_i is the i th column of Q .

Lemma 2.7.2. *Let $r > 0$ and let X be a random variable with χ_r distribution. Then as $r \rightarrow \infty$ the p.d.f. of $X - \sqrt{r}$ converges uniformly on any fixed interval to the p.d.f. of a normal distribution of mean 0 and variance 1/2.*

While both the β -Hermite and (β, a) -Laguerre ensembles are considered in detail in [DE05], only the main results for the β -Hermite ensemble will be presented in any detail here. The modifications to the (β, a) -Laguerre which are not straight-forward will be remarked upon in the appropriate place.

For the β -Hermite results that follow, let n be fixed and denote by $h_1^{(n)}, \dots, h_n^{(n)}$ the roots of the n th normalized Hermite polynomial $H_n(x)$. Let A_β be a matrix from the β -Hermite ensemble, rescaled by $1/\sqrt{2n\beta}$:

$$A_\beta = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2n\beta}} \begin{pmatrix} N(0, 2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0, 2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0, 2) & \chi_\beta \\ & & & \chi_\beta & N(0, 2) \end{pmatrix} \quad (2.49)$$

Let H be the $n \times n$ symmetric tridiagonal matrix which encodes the three-term recurrence for the

normalized Hermite polynomials.

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{n-1} & & & \\ \sqrt{n-1} & 0 & \sqrt{n-2} & & \\ & \sqrt{n-2} & 0 & & \\ & & & \ddots & \\ & & & 0 & \sqrt{1} \\ & & & \sqrt{1} & 0 \end{pmatrix} \quad (2.50)$$

The eigenvalues of H are the roots of the n th Hermite polynomial $H_n(x)$ and the eigenvector corresponding to the i th eigenvalue is given by

$$v_i = \begin{pmatrix} H_{n-1}(h_i^{(n)}) \\ -H_{n-2}(h_i^{(n)}) \\ \vdots \\ (-1)^{n-1} H_1(h_i^{(n)}) \\ (-1)^n H_0(h_i^{(n)}) \end{pmatrix} \quad (2.51)$$

Remark 7. In [DE05], the ± 1 's are omitted from the entries of the eigenvector. Technically speaking, in this situation Lemma 2.7.1 can't be used. However, because these entries become the coefficients of Gaussian random variables, this doesn't affect the end result.

Finally, let

$$Z = \begin{pmatrix} M_n & N_{n-1} & & & \\ N_{n-1} & M_{n-1} & N_{n-2} & & \\ & N_{n-2} & M_{n-2} & & \\ & & & \ddots & \\ & & & & M_2 & N_1 \\ & & & & N_1 & M_1 \end{pmatrix} \quad (2.52)$$

where $M_i \sim N(0, 1)$ and $N_i \sim N(0, 1/4)$, with all random variables being mutually independent, up to the symmetry.

Theorem 2.7.3. (3.1 of [DE05]) *Let $\lambda_i(A_\beta)$ be the i th largest eigenvalue of A_β , for any fixed $1 \leq i \leq n$. Then,*

$$\lim_{\beta \rightarrow \infty} \lambda_i(A_\beta) = \frac{1}{\sqrt{2n}} h_i^{(n)} \quad (2.53)$$

and

$$\lim_{\beta \rightarrow \infty} \sqrt{\beta} \left(\lambda_1(A_\beta) - \frac{1}{\sqrt{2n}} h_1^{(n)}, \lambda_2(A_\beta) - \frac{1}{\sqrt{2n}} h_2^{(n)}, \dots, \lambda_n(A_\beta) - \frac{1}{\sqrt{2n}} h_n^{(n)} \right) = \frac{1}{\sqrt{2n}} G \quad (2.54)$$

where $G \equiv (G_1, G_2, \dots, G_n)$ is a n -variate Gaussian with covariance matrix

$$\text{Cov}(G_i, G_j) = \frac{\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) H_l^2(h_j^{(n)}) + \sum_{l=0}^{n-2} H_{l+1}(h_i^{(n)}) H_l(h_i^{(n)}) H_{l+1}(h_j^{(n)}) H_l(h_j^{(n)})}{(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)})) (\sum_{l=0}^{n-1} H_l^2(h_j^{(n)}))}. \quad (2.55)$$

The convergence here is of p.d.f.'s, uniformly on any fixed interval in \mathbb{R}^n .

The proof of this is based on the following lemma.

Lemma 2.7.4. (3.2 of [DE05]) *With the definitions introduced above,*

$$\lim_{\beta \rightarrow \infty} \sqrt{2n\beta} A_\beta - \sqrt{\beta} H = Z. \quad (2.56)$$

This follows by applying Lemma 2.7.2 to the entries of the matrix $\sqrt{2n\beta} A_\beta - \sqrt{\beta} H$. Then, entry by entry,

$$\lim_{\beta \rightarrow \infty} \sqrt{2n\beta} \left(A_\beta - \frac{1}{\sqrt{2n}} H \right) = Z. \quad (2.57)$$

This is saying that the appropriately scaled and centered entries of A_β converge in distribution to Gaussian random variables. It then follows from the Skorohod Embedding Theorem (see [EK86]) that there exists a probability space on which all of the random variables are defined and such that the convergence is almost sure. From this, and the fact that eigenvalues are continuous functions of matrix entries, it follows that

$$\lambda_i(A_\beta) = \lambda_i \left(\frac{1}{\sqrt{2n}} H + \frac{1}{\sqrt{2n\beta}} Z \right) + o(1). \quad (2.58)$$

After applying (2.53) of Theorem 2.7.3 and Lemma 2.7.1 to this expression, the result is

$$\lambda_i(A_\beta) = \frac{1}{\sqrt{2n}} h_i^{(n)} + \frac{1}{\sqrt{2n\beta}} \frac{v_i^T Z v_i}{v_i^T v_i} + o(1). \quad (2.59)$$

Recalling what v_i and Z are leads to the following

$$\sqrt{\beta} \left(\lambda_i(A_\beta) - \frac{1}{\sqrt{2n}} h_i^{(n)} \right) = \frac{1}{\sqrt{2n}} \frac{\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) M_{l+1} + 2 \sum_{l=1}^{n-1} H_l(h_i^{(n)}) H_{l-1}(h_i^{(n)}) N_l}{\sum_{l=0}^{n-1} H_l^2(h_i^{(n)})} + o(1). \quad (2.60)$$

This means that the scaled and centered random variable on the left-hand side is Gaussian with mean zero and variance σ^2 , with

$$\sigma^2 = \frac{1}{2n} \frac{\sum_{l=0}^{n-1} H_l^4(h_i^{(n)}) + \sum_{l=1}^{n-1} H_l^2(h_i^{(n)}) H_{l-1}^2(h_i^{(n)})}{\left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2} \quad (2.61)$$

Note that in going from (2.60) to (2.61), the 2 in front of the second sum is absorbed into the N_l 's, since these are Gaussian random variables with variance 1/4.

Consider now the vector

$$\sqrt{\beta} \left(\lambda_1(A_\beta) - \frac{1}{\sqrt{2n}} h_1^{(n)}, \dots, \lambda_n(A_\beta) - \frac{1}{\sqrt{2n}} h_n^{(n)} \right). \quad (2.62)$$

The same reasoning as used above leads to the following generalization:

$$\lim_{\beta \rightarrow \infty} \sqrt{\beta} \left(\lambda_1(A_\beta) - \frac{1}{\sqrt{2n}} h_1^{(n)}, \dots, \lambda_n(A_\beta) - \frac{1}{\sqrt{2n}} h_n^{(n)} \right) = \frac{1}{\sqrt{2n}} G \quad (2.63)$$

where G is a n -variate Gaussian with covariance matrix Σ given by

$$\Sigma_{ij} = \frac{\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) H_l^2(h_j^{(n)}) + \sum_{l=0}^{n-2} H_{l+1}(h_i^{(n)}) H_l(h_i^{(n)}) H_{l+1}(h_j^{(n)}) H_l(h_j^{(n)})}{\left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right) \left(\sum_{l=0}^{n-1} H_l^2(h_j^{(n)}) \right)}. \quad (2.64)$$

Chapter 3

Large β : Hermite Ensemble

3.1 Introduction

After a section on the general set-up for the work that follows, LLN and CLT results for the eigenvalues will be established. For a $n \times n$ matrix, this will give a n -dimensional vector of random eigenvalues which will be shown to be Gaussian. The covariance matrix will be computed in terms of sums of Hermite polynomials. Then we will take $n \rightarrow \infty$ and compute the continuum limit of the finite sums. This will turn out to involve integrals of Airy functions. The asymptotics of this infinite dimensional covariance matrix will be investigated. Finally, it will be shown that the limits in terms of the parameter β and the matrix size n commute. That is, the following diagram commutes:

$$\begin{array}{ccc} \beta < \infty, n < \infty & \xrightarrow{n \rightarrow \infty} & \text{Theorem 2.5.1} \\ \beta \rightarrow \infty \downarrow & & \downarrow \beta \rightarrow \infty \\ \text{Theorem 2.7.3} & \xrightarrow{n \rightarrow \infty} & \text{Theorem 3.2.2} \end{array}$$

3.2 β -Hermite as $\beta \rightarrow \infty$

The main results of this chapter are presented. Throughout this chapter, ϵ_n will denote a quantity that goes to zero as $n \rightarrow \infty$.

Theorem 3.2.1. (Law of Large Numbers) *Let A_β be a matrix from the $n \times n$ β -Hermite ensemble, scaled by $1/\sqrt{2n\beta}$, and let $\lambda_i(A_\beta)$ be the i th largest eigenvalue of A_β . Then*

$$\lim_{n \rightarrow \infty} \lim_{\beta \rightarrow \infty} n^{2/3}(\lambda_i(A_\beta) - 1) = \frac{a_i}{3^{1/3}2}, \quad (3.1)$$

where a_i is the i th largest root of the Airy function, $Ai(x)$.

Proof. By Theorem 2.7.3, $\lim_{\beta \rightarrow \infty} \lambda_i(A_\beta) = \frac{1}{\sqrt{2n}} h_i^{(n)}$, so using the following asymptotic expansion for $h_i^{(n)}$ (see [Sze03]),

$$h_i^{(n)} \rightarrow (2n+1)^{1/2} + 6^{-1/3}(2n+1)^{-1/6}a_i \quad (3.2)$$

leads to

$$\frac{1}{\sqrt{2n}} h_i = \frac{1}{\sqrt{2n}} ((2n+1)^{1/2} + 6^{-1/3}(2n+1)^{-1/6}a_i) \quad (3.3)$$

$$= \frac{(2n+1)^{1/2}}{(2n)^{1/2}} + \frac{a_i}{6^{1/3}(2n+1)^{1/6}(2n)^{1/2}}. \quad (3.4)$$

The last expression above becomes

$$\frac{1}{\sqrt{2n}} h_i = 1 + \frac{a_i}{6^{1/3}2^{2/3}n^{2/3}} + \epsilon_n. \quad (3.5)$$

Combining these results yields

$$\lambda_i(A_\beta) = 1 + \frac{a_i}{3^{1/3}2n^{2/3}} + \epsilon_n, \quad (3.6)$$

and the proof follows. □

Theorem 3.2.2. (Central Limit Theorem) *Let A_β be a matrix from the $n \times n$ β -Hermite ensemble, scaled by $1/\sqrt{2n\beta}$, and let $\lambda_i(A_\beta)$ be the i th largest eigenvalue of A_β . Then,*

$$\lim_{n \rightarrow \infty} \lim_{\beta \rightarrow \infty} n^{2/3} \sqrt{\beta} \left(\lambda_1(A_\beta) - \frac{1}{\sqrt{2n}} h_1^{(n)}, \lambda_2(A_\beta) - \frac{1}{\sqrt{2n}} h_2^{(n)}, \dots, \lambda_n(A_\beta) - \frac{1}{\sqrt{2n}} h_n^{(n)} \right) = \tilde{G}, \quad (3.7)$$

where $\tilde{G} \equiv (\tilde{G}_1, \dots, \tilde{G}_n, \dots)$ is an infinite Gaussian random vector with covariance matrix $\tilde{\Sigma}$, with

$$\tilde{\Sigma}_{ij} = \text{Cov}(\tilde{G}_i, \tilde{G}_j) = \frac{\int_0^\infty Ai^2(x+a_i) Ai^2(x+a_j) dx}{(\int_0^\infty Ai^2(x+a_i) dx)(\int_0^\infty Ai^2(x+a_j) dx)}. \quad (3.8)$$

Since the proof of (3.7) relies on (3.8), the covariance result will be proved first. The proof is quite long, so it will be broken into smaller pieces, as follows.

(1) Convergence of the first sum in the numerator, $i = j$:

$$\lim_{n \rightarrow \infty} \sum_{l=0}^{n-1} H_l^4(h_i^{(n)}) = \int_0^\infty Ai^4(x + a_i) dx. \quad (3.9)$$

(2) Convergence of the second sum in the numerator, $i = j$:

$$\lim_{n \rightarrow \infty} \sum_{l=0}^{n-2} H_{l+1}^2(h_i^{(n)}) H_l^2(h_i^{(n)}) = \int_0^\infty Ai^4(x + a_i) dx. \quad (3.10)$$

(3) Convergence of the sum in the denominator, $i = j$:

$$\lim_{n \rightarrow \infty} \left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2 = \left(\int_0^\infty Ai^2(x + a_i) dx \right)^2. \quad (3.11)$$

(4) Convergence of the ratio of the sums when scaled by $n^{2/3}$, $i = j$:

$$\lim_{n \rightarrow \infty} n^{2/3} \frac{\sum_{l=0}^{n-1} H_l^4(h_i^{(n)}) + \sum_{l=0}^{n-2} H_{l+1}^2(h_i^{(n)}) H_l^2(h_i^{(n)})}{\left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2} = \frac{\int_0^\infty Ai^4(x + a_i) dx}{\left(\int_0^\infty Ai^2(x + a_i) dx \right)^2}. \quad (3.12)$$

(5) Finally, it will be shown that the case $i \neq j$ involves only a slight modification of the preceding argument, thereby completing the calculation of all terms of the covariance matrix.

The starting point in the derivation of the covariance matrix is the expression,

$$Cov(G_i, G_j) = \frac{\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) H_l^2(h_j^{(n)}) + \sum_{l=0}^{n-2} H_{l+1}(h_i^{(n)}) H_l(h_i^{(n)}) H_{l+1}(h_j^{(n)}) H_l(h_j^{(n)})}{\left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right) \left(\sum_{l=0}^{n-1} H_l^2(h_j^{(n)}) \right)}. \quad (3.13)$$

When $i = j$, this simplifies to

$$Var(G_i) = Cov(G_i, G_i) = \frac{\sum_{l=0}^{n-1} H_l^4(h_i^{(n)}) + \sum_{l=0}^{n-2} H_{l+1}^2(h_i^{(n)}) H_l^2(h_i^{(n)})}{\left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2} \quad (3.14)$$

Proof of (1): The roots of Hermite polynomials have asymptotic expansion, (see [Sze03]),

$$h_i^{(n)} = (2n+1)^{1/2} + 6^{-1/3}(2n+1)^{-1/6}a_i + \epsilon_n. \quad (3.15)$$

The terms under consideration are the l th Hermite polynomials evaluated at roots of the n th Hermite polynomial. This means that to use the above asymptotics, $h_i^{(n)}$ must be rewritten in terms of $h_i^{(l)}$ and Δ_l^n , where Δ_l^n is a correction term. Explicitly,

$$h_i^{(n)} = (2n+1)^{1/2} + 6^{-1/3}(2n+1)^{-1/6}a_i + O(\epsilon_n) = (2l+1)^{1/2} - 2^{-1/2}3^{-1/3}l^{-1/6}(a_i + \Delta_l^n), \quad (3.16)$$

from which it follows that

$$a_i + \Delta_l^n = \frac{(2n+1)^{1/2} - (2l+1)^{1/2} + 6^{-1/3}(2n+1)^{-1/6}a_i}{2^{-1/2}3^{-1/3}l^{-1/6}}. \quad (3.17)$$

Using the Plancherel-Rotach asymptotic estimate for $H_l(x)$ (see [Sze03]),

$$e^{-x^2/2}H_l(x) = \frac{3^{1/3}2^{1/4}}{\pi}l^{-1/12}(Ai(t) + O(l^{-2/3})) \quad (3.18)$$

gives

$$e^{-(h_i^{(n)})^2/2}H_l(h_i^{(n)}) = c \times l^{-1/12}Ai(a_i + \Delta_l^n). \quad (3.19)$$

This leads to

$$H_l^2(h_i^{(n)}) = c \times e^{(h_i^{(n)})^2}l^{-1/6}Ai^2(a_i + \Delta_l^n), \quad (3.20)$$

$$H_l^4(h_i^{(n)}) = c \times e^{2(h_i^{(n)})^2}l^{-1/3}Ai^4(a_i + \Delta_l^n), \quad (3.21)$$

where c is a constant that doesn't depend on l , different in each line. For the time being, ignore the $c \times e^{2(h_i^{(n)})^2}$; it needlessly complicates the notation and it will be addressed below.

Since the above asymptotic estimates only hold in the regime where l is close to n , the sum should be broken into two pieces, one for large l and the other for small l . More precisely, write

$$\sum_{l=0}^{n-1} H_l^4(h_i^{(n)}) = \sum_{l=0}^{n-Cn^{1/3}} H_l^4(h_i^{(n)}) + \sum_{l=n-Cn^{1/3}+1}^{n-1} H_l^4(h_i^{(n)}). \quad (3.22)$$

Changing variables, $l = n - m$, gives

$$\sum_{l=0}^{n-1} H_l^4(h_i^{(n)}) = \sum_{m=1}^{Cn^{1/3}} H_l^4(h_i^{(n)}) + \sum_{m=Cn^{1/3}+1}^n H_l^4(h_i^{(n)}). \quad (3.23)$$

Notice that the first sum now corresponds to values that are near n .

It will now be shown that

$$\lim_{n \rightarrow \infty} \sum_{m=1}^{Cn^{1/3}} (n-m)^{-1/3} Ai^4(a_i + \Delta_{n-m}^k) = \int_0^C Ai^4(a_i + x) dx, \quad (3.24)$$

and

$$\lim_{n \rightarrow \infty} \sum_{m=Cn^{1/3}+1}^n (n-m)^{-1/3} Ai^4(a_i + \Delta_{n-m}^n) = 0. \quad (3.25)$$

Letting $C \rightarrow \infty$ will complete the derivation of the integrals.

Since the transition is from sums to integrals, it will be helpful to recall the definition of a Riemann sum ([BO99]):

$$\lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} f(\bar{t}_n)(t_{n+1} - t_n) = \int_a^b f(t) dt,$$

where $f(t)$ is continuous, \bar{t}_n is any point in the interval $t_n \leq \bar{t}_n \leq t_{n+1}$, and $t_n = a + n(b-a)/N$.

In what follows, $f(t) = Ai^4(t)$, $t_n = n/k^{1/3}$, so $t_{n+1} - t_n = 1/k^{1/3}$, $N = Ck^{1/3}$, and $a = 0, b = C$.

With the current notation, this becomes

$$\lim_{n \rightarrow \infty} \sum_{m=1}^{Cn^{1/3}} (n-m)^{-1/3} Ai^4(a_i + \Delta_{n-m}^n) = \lim_{n \rightarrow \infty} \sum_{m=1}^{Cn^{1/3}} (n)^{-1/3} Ai^4(a_i + \Delta_{n-m}^n) + o(\epsilon_n), \quad (3.26)$$

since in this regime $(n-m)^{-1/3} = n^{-1/3}(1 + m/n + O(n^{-2}))$ and $m/n \rightarrow 0$ as $n \rightarrow \infty$.

To complete the criteria for a Riemann sum, it must be shown that $\Delta_{n-m}^n = O(mn^{-1/3})$.

Now,

$$\Delta_{n-m}^n = \frac{(2n+1)^{1/2} - (2(n-m)+1)^{1/2} + 6^{-1/3}(2n+1)^{-1/6}a_i}{2^{-1/2}3^{-1/3}(n-m)^{-1/6}} - a_i \quad (3.27)$$

$$= \frac{(2n+1)^{1/2} - (2(n-m)+1)^{1/2} + 6^{-1/3}(2n+1)^{-1/6}a_i - 2^{-1/2}3^{-1/3}(n-m)^{-1/6}a_i}{2^{-1/2}3^{-1/3}(n-m)^{-1/6}} \quad (3.28)$$

$$= n^{1/6}((2n+1)^{1/2} - (2n-2m+1)^{1/2}) + o(\epsilon_n), \quad (3.29)$$

since $6^{-1/3}(2n+1)^{-1/6} = 2^{-1/2}3^{-1/3}(n+1/2)^{-1/6}$. This gives

$$\Delta_{n-m}^n = n^{1/6}((2n+1)^{1/2} - (2n-2m+1)^{1/2}) \quad (3.30)$$

$$= n^{2/3} \left(\left(1 + \frac{1}{2n}\right)^{1/2} - \left(1 + \frac{1}{2n} - \frac{m}{n}\right)^{1/2} \right) \quad (3.31)$$

$$= n^{2/3} \frac{m}{n} + O(n^{-4/3}) = mn^{-1/3} + o(\epsilon_n). \quad (3.32)$$

using the Binomial expansion truncated at the terms of order $O(n^{-2})$. Since m runs from 1 to $Cn^{1/3}$, the term Δ_{n-m}^n is \bar{t}_m .

Putting this all together,

$$\lim_{n \rightarrow \infty} \sum_{m=1}^{Cn^{1/3}} n^{-1/3} Ai^4(a_i + \Delta_{n-m}^n) = \int_0^C Ai^4(a_i + x) dx.$$

The final step is to let $C \rightarrow \infty$.

Now it must be shown that the other sum decays to 0. In this sum, l is small compared to n , or in terms of the other index, m is large. If l is small Δ_l^n is large, so $Ai(a_i + \Delta_l^n)$ looks like $Ai(x)$ as $x \rightarrow \infty$. The asymptotics for the Airy function as $x \rightarrow \infty$ is given by (see [BO99])

$$Ai(x) \approx \frac{1}{2} \pi^{-1/2} x^{-1/4} e^{-(2/3)x^{3/2}}, \quad (3.33)$$

so

$$Ai^4(x) \approx \frac{1}{16\pi^2} \frac{1}{x} e^{-(8/3)x^{3/2}}. \quad (3.34)$$

Then in this regime, the limiting sum is

$$\lim_{n \rightarrow \infty} \sum_{l=1}^{n-Cn^{1/3}} \frac{1}{l^{1/3}} \frac{1}{l^{1/6}((2n+1)^{1/2} - (2l+1)^{1/2})} e^{-(8/3)(l^{1/6}((2n+1)^{1/2} - (2l+1)^{1/2}))}. \quad (3.35)$$

Since l is small compared to n , this sum is asymptotically

$$\lim_{n \rightarrow \infty} e^{-n^{3/4}} \sum_{l=1}^{n-Cn^{1/3}} \frac{1}{l^{1/2}((2n+1)^{1/2} - (2l+1)^{1/2})}, \quad (3.36)$$

Replace the summand by the term which maximizes it and sum over that constant term. The sum will only contribute something that is polynomial in n , while the term out front is exponential in $-n$. This exponential term will dominate the polynomial term and the limit will be zero.

□

Proof of (2): A modification of the above argument can be used on the other sum in the numerator.

Consider

$$\lim_{n \rightarrow \infty} \sum_{l=0}^{n-2} H_{l+1}^2(h_i^{(n)}) H_l^2(h_i^{(n)}) \quad (3.37)$$

The asymptotics give

$$H_{l+1}^2(h_i^{(n)}) = (l+1)^{-1/6} Ai^2(a_i + \Delta_{l+1}^n) \Rightarrow (n-m-1)^{-1/6} Ai^2(a_i + \Delta_{n-m-1}^n) + O(\epsilon_n), \quad (3.38)$$

and

$$H_l^2(h_i^{(n)}) = (l)^{-1/6} Ai^2(a_i + \Delta_l^n) \Rightarrow (n-m)^{-1/6} Ai^2(a_i + \Delta_{n-m}^n) + O(\epsilon_n). \quad (3.39)$$

Putting these together gives

$$\lim_{n \rightarrow \infty} \sum_{m=0}^{Cn^{1/3}} (n-m-1)^{-1/6} Ai^2(a_i + (m+1)n^{-1/3}) (n-m)^{-1/6} Ai^2(a_i + mn^{-1/3}). \quad (3.40)$$

In the regime of interest, $\lim_{n \rightarrow \infty} (n-m-1)/n = 1$ and the $a_i + (m+1)n^{-1/3}$ corresponds to the right endpoint of the Riemann sum interval and the $a_i + mn^{-1/3}$ term corresponds to the left endpoint. Since the location of the points in the interval in Riemann sums doesn't matter, this is equivalent to

$$\lim_{n \rightarrow \infty} \sum_{m=0}^{Cn^{1/3}} n^{-1/3} Ai^2(a_i + \bar{m}) Ai^2(a_i + \bar{m}) = \int_0^C Ai^4(x + a_i) dx. \quad (3.41)$$

□

Proof of (3): For the denominator, the argument must be modified since the sums are of the form

$$\left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2, \quad (3.42)$$

and the summand only involves the product of two Hermite polynomials. This will change the scaling in the Riemann sum. First, since $x \rightarrow x^2$ is a continuous function,

$$\lim_{n \rightarrow \infty} \left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2 = \left(\lim_{n \rightarrow \infty} \sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2. \quad (3.43)$$

Using the same asymptotics as above, except for $H_l^2(h_i^{(n)})$ instead of $H_l^4(h_i^{(n)})$, leads to

$$\left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2 = \left(\sum l^{-1/6} A i^2 (a_i + \Delta_l^n) \right)^2. \quad (3.44)$$

Again, the only contribution to the Riemman sum comes from the regime where $l \sim n$, so this can be written as

$$\left(\sum_{l=n-Cn^{1/3}}^n H_l^2(h_i^n) \right)^2 = \left(\sum_{l=n-Cn^{1/3}}^n l^{-1/6} A i^2 (a_i + \Delta_l^n) \right)^2 \quad (3.45)$$

$$= \left(\sum_{l=n-Cn^{1/3}}^n n^{-1/6} A i^2 (a_i + \Delta_l^n) \right)^2 + O(\epsilon_n). \quad (3.46)$$

Since $\Delta_l^n = O(n^{-1/3})$, the above sum doesn't have the right scaling. To address this, multiply and divide by $n^{-1/6}$.

$$\left(\sum_{l=n-Cn^{1/3}}^n n^{1/6} n^{-1/6} n^{-1/6} A i^2 (a_i + \Delta_l^n) \right)^2 = n^{1/3} \left(\sum_{l=n-Cn^{1/3}}^n n^{-1/3} A i^2 (a_i + \Delta_l^n) \right)^2. \quad (3.47)$$

Now the sum will converge to the appropriate integral. The extra $n^{1/3}$ term will be dealt with next.

□

Proof of (4): In the Plancherel-Rotach asymptotic expansion (3.18), there is a term of the form $c \times e^{2(h_i^{(n)})^2}$. The same constant arises in both sums in the numerator and the sum in the denominator.

Since this term doesn't depend on l it can be brought out of all of the sums, and they cancel out.

This is why they have been omitted in the preceding calculations.

Recall that we are trying to prove

$$\lim_{n \rightarrow \infty} \lim_{\beta \rightarrow \infty} n^{2/3} \sqrt{\beta} \left(\lambda_1(A_\beta) - \frac{1}{\sqrt{2n}} h_1^{(n)}, \lambda_2(A_\beta) - \frac{1}{\sqrt{2n}} h_2^{(n)}, \dots, \lambda_n(A_\beta) - \frac{1}{\sqrt{2n}} h_n^{(n)} \right) = G, \quad (3.48)$$

By Theorem 2.7.3 on page 22, the β limit leads to

$$\lim_{\beta \rightarrow \infty} \sqrt{\beta} \left(\lambda_1(A_\beta) - \frac{1}{\sqrt{2n}} h_1^{(n)}, \lambda_2(A_\beta) - \frac{1}{\sqrt{2n}} h_2^{(n)}, \dots, \lambda_n(A_\beta) - \frac{1}{\sqrt{2n}} h_n^{(n)} \right) = \frac{1}{\sqrt{2n}} G. \quad (3.49)$$

Substituting the right-hand side into the expression containing the n limit gives

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2}} \frac{n^{2/3}}{n^{1/2}} G = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2}} n^{1/6} G \quad (3.50)$$

Since G is Gaussian, the $n^{1/6}/\sqrt{2}$ is squared when calculating the variance of $(n^{1/6}/\sqrt{2})G \equiv \tilde{G}$.

This then gives the following expression for the variance at finite n :

$$Var(\tilde{G}_i) = Cov(\tilde{G}_i, \tilde{G}_i) = \frac{n^{1/3}}{2} \frac{\sum_{l=0}^{n-1} H_l^4(h_i^{(n)}) + \sum_{l=0}^{n-2} H_{l+1}^2(h_i^{(n)}) H_l^2(h_i^{(n)})}{\left(\sum_{l=0}^{n-1} H_l^2(h_i^{(n)}) \right)^2}. \quad (3.51)$$

From the work above, this becomes

$$\frac{n^{1/3}}{2} \frac{2 \sum n^{-1/3} A i^4 (a_i + \Delta_l^n)}{n^{1/3} \left(\sum n^{-1/3} A i^2 (a_i + \Delta_l^n) \right)^2}. \quad (3.52)$$

The extra $n^{1/3}$ in the denominator that was the byproduct of the Riemann sum scaling is canceled by the $n^{1/3}$ in the numerator. Taking the limit then gives

$$\lim_{n \rightarrow \infty} \frac{n^{1/3}}{2} \frac{2 \sum n^{-1/3} A i^4 (a_i + \Delta_l^n)}{n^{1/3} \left(\sum n^{-1/3} A i^2 (a_i + \Delta_l^n) \right)^2} = \lim_{n \rightarrow \infty} \frac{\sum n^{-1/3} A i^4 (a_i + \Delta_l^n)}{\left(\sum n^{-1/3} A i^2 (a_i + \Delta_l^n) \right)^2}, \quad (3.53)$$

but the proofs of (1), (2), and (3) above show that

$$\lim_{n \rightarrow \infty} \frac{\sum n^{-1/3} A i^4 (a_i + \Delta_l^n)}{\left(\sum n^{-1/3} A i^2 (a_i + \Delta_l^n) \right)^2} = \frac{\int_0^\infty A i^4 (x + a_i) dx}{\left(\int_0^\infty A i^2 (x + a_i) dx \right)^2} \quad (3.54)$$

This completes the proof for $i = j$ □

Proof of (5): Finally, turn to the situation $i \neq j$. The only difference is that at the there are now two different Hermite roots, $h_i^{(n)}$ and $h_j^{(n)}$. All of the asymptotics used before can still be used, except that now there will be two Airy roots, a_i and a_j . Everything with the Riemann sums is exactly the same, up to a change of indices in the appropriate places. So, the end result is

$$Cov(\tilde{G}_i, \tilde{G}_j) = \frac{\int_0^\infty A i^2 (x + a_i) A i^2 (x + a_j) dx}{\left(\int_0^\infty A i^2 (x + a_i) dx \right) \left(\int_0^\infty A i^2 (x + a_j) dx \right)}, \quad (3.55)$$

and this completes the proof for all entries of the covariance matrix. □

Proof of (3.7). To prove that two infinite Gaussian vectors are the same in law, it is enough to prove equality in law for any finite subcollection. Since the components are Gaussian, this means showing that the mean and covariance are equal. To do this, take any k entries from the left-hand side of (3.7), and the corresponding entries from the right hand side. Denote these as

$$G_k = \left(\lambda_{i_1}(A_\beta) - \frac{1}{\sqrt{2n}} h_{i_1}^{(n)}, \lambda_{i_2}(A_\beta) - \frac{1}{\sqrt{2n}} h_{i_2}^{(n)}, \dots, \lambda_{i_k}(A_\beta) - \frac{1}{\sqrt{2n}} h_{i_k}^{(n)} \right) \quad (3.56)$$

$$\tilde{G}_k = (\tilde{G}_{i_1}, \dots, \tilde{G}_{i_k}). \quad (3.57)$$

Then, by Theorem 2.7.3, the covariance of (3.56) is given by a $k \times k$ matrix of sums of Hermite polynomials evaluated at $h_{i_j}^{(n)}$ for $j = 1, \dots, k$. By the proof of (3.8), these sums converge to integrals of Airy functions as $n \rightarrow \infty$. But that is precisely the covariance structure for (3.57) by definition. Since both of the vectors have mean zero, they have the same distribution. Since this holds for any arbitrary finite collection, the proof is complete. □

Now the asymptotics of the covariance matrix will be computed.

Theorem 3.2.3. (Asymptotics of Covariance Matrix) *With the same assumptions as above, for i fixed and $j \rightarrow \infty$,*

$$\text{Cov}(G_i, G_j) = O(j^{-2/3}). \quad (3.58)$$

Along the main diagonal, as $i \rightarrow \infty$,

$$\text{Var}(G_i) = \text{Cov}(G_i, G_i) = O(i^{-1/3}). \quad (3.59)$$

The proofs of these results will be presented separately.

Proof of (3.58). For a fixed i , there are i roots of $Ai^2(x + a_i)$ along the positive x-axis, including the one at $x = 0$. As $j \rightarrow \infty$, $Ai^2(x + a_j)$ becomes highly oscillatory, but the oscillations are confined to lie under an appropriate scaled version of $Ai^2(x + a_i)$. Since the Airy function decays (see [OLBC10]) as

$$Ai(z) = \frac{e^{-\frac{2}{3}z^{3/2}}}{z^{1/4}} + o(z^{2/3}), \quad (3.60)$$

the only portion of the integrand that will make a contribution to the value of the integral lies between the origin and the beginning of the exponential decay. See figures 3.1.

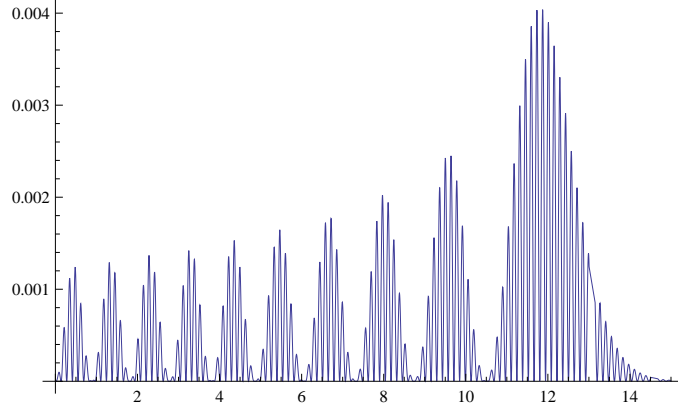


Figure 3.1: Graph of $\text{Ai}^2(x + a_{10})\text{Ai}^2(x + a_{2500})$

The question now is to find the appropriate scaling. The behavior of a_j is given by (see [OLBC10]),

$$a_j = -T \left(\frac{3}{8}\pi(4j - 1) \right) \quad (3.61)$$

where

$$T(t) = t^{\frac{2}{3}} \left(1 + \frac{5}{48}t^{-2} - \frac{5}{36}t^{-4} + \dots \right) \quad (3.62)$$

As $j \rightarrow \infty$, the asymptotic behavior of $\text{Ai}^2(x + a_j)$ is given by

$$\frac{1}{\pi(x + a_j)^{\frac{1}{2}}} \sin^2 \left(\frac{2}{3}(x + a_j)^{\frac{3}{2}} + \frac{1}{4}\pi \right), \quad (3.63)$$

but since $x \ll a_j$ in the region under consideration due to the exponential decay, as $j \rightarrow \infty$, the $x + a_j$ terms can be replaced by the first order term of the asymptotic expansion for a_j . This leads to

$$\text{Ai}^2(x + a_j) = \frac{1}{\pi \left(\left(\frac{3}{8}\pi(4j - 1) \right)^{\frac{2}{3}} \right)^{\frac{1}{2}}} \sin^2 \left(\frac{2}{3}(x + a_j)^{\frac{3}{2}} + \frac{1}{4}\pi \right) \quad (3.64)$$

$$= \frac{1}{\pi \left(\frac{3}{8}\pi(4j - 1) \right)^{\frac{1}{3}}} \sin^2 \left(\frac{2}{3}(x + a_j)^{\frac{3}{2}} + \frac{1}{4}\pi \right) \quad (3.65)$$

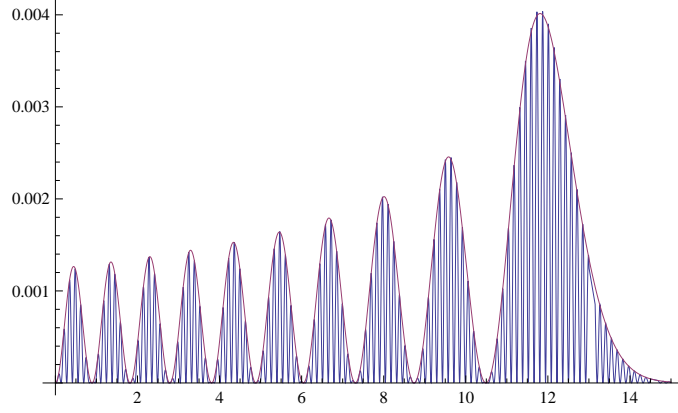


Figure 3.2: Comparison of scaled $\text{Ai}^2(x + a_{10})$ and $\text{Ai}^2(x + a_{10})\text{Ai}^2(x + a_{2500})$

Since the sine term is highly oscillatory in j and the prefactor term decays in j , the scaling should be chosen to be this prefactor term, see figure 3.2.

Due to the highly oscillatory behavior of the integrand in j , the area under the graph of this function is bounded between the maximum and minimum values of the \sin^2 term. This means that the total area under the graph is bounded above by the scaling term. This leads to the following estimate

$$\frac{\int_0^\infty \text{Ai}^2(x + a_i)\text{Ai}^2(x + a_j)dx}{(\int_0^\infty \text{Ai}^2(x + a_i)dx)(\int_0^\infty \text{Ai}^2(x + a_j)dx)} \leq \frac{\frac{1}{\pi(\frac{3}{8}\pi(4j-1))^{\frac{1}{3}}} \int_0^\infty \text{Ai}^2(x + a_i)dx}{(\int_0^\infty \text{Ai}^2(x + a_i)dx)(\int_0^\infty \text{Ai}^2(x + a_j)dx)} \quad (3.66)$$

$$= \frac{\frac{1}{\pi(\frac{3}{8}\pi(4j-1))^{\frac{1}{3}}}}{(\int_0^\infty \text{Ai}^2(x + a_j)dx)} \quad (3.67)$$

$$= \frac{1}{\pi(\frac{3}{8}\pi(4j-1))^{\frac{1}{3}}(\text{Ai}'(a_j))^2} \quad (3.68)$$

To complete the asymptotic analysis, the behavior of $\text{Ai}'(a_j)$ needs to be understood. Using the asymptotic expansion (see [OLBC10])

$$\text{Ai}'(a_j) = (-1)^{j-1}V\left(\frac{3}{8}\pi(4j-1)\right) \quad (3.69)$$

where

$$V(t) = \pi^{-\frac{1}{2}}t^{\frac{1}{6}}\left(1 + \frac{5}{48}t^{-2} - \frac{1525}{4608}t^{-4} - \dots\right) \quad (3.70)$$

leads to the following first order approximation:

$$(Ai'(a_j))^2 \approx \pi^{-1} \left(\frac{3}{8} \pi (4j - 1) \right)^{\frac{1}{3}} \quad (3.71)$$

Putting this into (3.68) gives

$$\sigma_{ij}^2 \leq \frac{1}{\left(\frac{3}{8} \pi (4j - 1) \right)^{\frac{2}{3}}} \quad (3.72)$$

as $j \rightarrow \infty$, which completes the proof. □

Proof of (3.59). Using the first mean value theorem for infinite integrals (see [GR94]), gives

$$\frac{\int_0^\infty Ai^4(x + a_i) dx}{\left(\int_0^\infty Ai^2(x + a_i) dx \right)^2} = \frac{\int_{a_i}^\infty Ai^2(x) Ai^2(x) dx}{\left(\int_{a_i}^\infty Ai^2(x) dx \right)^2} \quad (3.73)$$

$$= \frac{\mu \int_{a_i}^\infty Ai^2(x) dx}{\left(\int_{a_i}^\infty Ai^2(x) dx \right)^2} \quad (3.74)$$

$$= \frac{\mu}{\int_{a_i}^\infty Ai^2(x) dx} \quad (3.75)$$

$$= \frac{\mu}{(Ai'(a_i))^2} \quad (3.76)$$

where $m \leq \mu \leq M$, with m, M the minimum and maximum value of $Ai^2(x + a_i)$ over $[0, \infty)$.

Using again (3.71), this is

$$\sigma_{ii}^2 = \frac{\mu}{(Ai'(a_i))^2} \quad (3.77)$$

$$= \frac{\mu}{\pi^{-1} \left(\frac{3}{8} \pi (4i - 1) \right)^{\frac{1}{3}}} + o(i^{-11/6}) \quad (3.78)$$

$$= \frac{c}{(4i - 1)^{\frac{1}{3}}} + o(i^{-11/6}) \quad (3.79)$$

as $i \rightarrow \infty$, which completes the proof. □

3.3 Commuting the limits

The main result of this section establishes that the β and n limits commute.

Theorem 3.3.1. *Let $\epsilon = 2/\sqrt{\beta}$. Let $(\Lambda_\epsilon, f_\epsilon)$ be the eigen-pair corresponding to ϵ , and let (Λ, f) be the eigen-pair corresponding to $\epsilon = 0$. Then,*

$$\lim_{\epsilon \rightarrow 0} \frac{\Lambda_\epsilon - \Lambda}{\epsilon} = \int_0^\infty f^2 db. \quad (3.80)$$

Switching from ϵ back to $\frac{2}{\sqrt{\beta}}$, this limit becomes

$$\lim_{\beta \rightarrow \infty} \frac{\sqrt{\beta}(\Lambda_{2/\sqrt{\beta}} - \Lambda)}{2} = \int_0^\infty f^2 db. \quad (3.81)$$

The right hand side of (3.81) is a Gaussian random variable with mean zero and variance

$$\int_0^\infty (f^2)^2 dx = \int_0^\infty f^4(x) dx \quad (3.82)$$

Recalling that $f(x)$ is the normalized Airy function, this is

$$\frac{\int_0^\infty Ai^4(x + a_1) dx}{\left(\int_0^\infty Ai^2(x + a_1) dx\right)^2} \quad (3.83)$$

This is the same expression that was found by taking $\beta \rightarrow \infty$ and then $n \rightarrow \infty$, establishing that the limits commute.

Remark 8. *Dumitriu and Edelman's scaling differs from Ramírez, Rider, and Virág's by a factor of 1/2. This is why the 2 shows up in the denominator of the expression above. This makes the two scalings the same.*

Proof. Define

$$\Gamma_\epsilon(f, g) = \int_0^\infty f'(x)g'(x)dx + \int_0^\infty xf(x)g(x)dx + \epsilon \int_0^\infty f(x)g(x)b'(x)dx \quad (3.84)$$

with $\epsilon = \frac{2}{\sqrt{\beta}}$.

Recall that a pair $(\Lambda_\epsilon, f_\epsilon)$ is an eigen-pair if it satisfies

$$\Lambda_\epsilon \int_0^\infty \varphi(x)f_\epsilon(x)dx = \Gamma_\epsilon(\varphi, f_\epsilon) \quad (3.85)$$

for any $\varphi \in C_0^\infty$.

The variational characterization is given in terms of the bilinear form $\langle \cdot, H \cdot \rangle$ on $C_0^\infty \times L^*$.

In what follows, this bilinear form needs to be extended to act on $L^* \times L^*$. But this is exactly

the content of Proposition 2.4 in [RR09]. With this bilinear form now extended to $L^* \times L^*$, the variational characterization of eigen-pairs, (3.85) holds for any $\varphi \in L^*$.

Consider the family $\{f_\epsilon\}$ with $\epsilon \rightarrow 0$. For each value of ϵ , $f_\epsilon \in L^*$ and is bounded, since it is an eigenfunction. This means that the entire family is in bounded in L^* , so there exists a subsequence that converges to some $f \in L^*$ in the following ways: $f_\epsilon \rightarrow f$ in L^2 , $f'_\epsilon \rightarrow f'$ weakly in L^2 , $f_\epsilon \rightarrow f$ uniformly on compact sets, and $f_n \rightarrow f$ weakly in L^* .

Using these modes of convergence,

$$\langle f_\epsilon, f_\epsilon \rangle_{H_\epsilon} = \|f_\epsilon\|_*^2 - \|f_\epsilon\|_2^2 - \epsilon \left(\int_0^\infty F(f_\epsilon, b_x) dx \right) \quad (3.86)$$

$$\Rightarrow \|f\|_*^2 - \|f\|_2^2 \quad (3.87)$$

with the convergence being weakly in L^* . $F(f_\epsilon, b_x)$ is a function of f_ϵ and b_x such that for any $\delta > 0$, there exists a random variable X such that

$$\left| \epsilon \left(\int_X^\infty F(f_\epsilon, b_x) dx \right) \right| \leq \delta \|f_\epsilon\|_*. \quad (3.88)$$

Now it will be shown that the limiting function f is the eigenfunction corresponding to $\epsilon = 0$. Assume that f is not an eigenfunction, but that f_0 is. Then by the variational definition,

$$\Lambda = \langle f_0, f_0 \rangle_{H_0} \leq \langle f, f \rangle_{H_0}. \quad (3.89)$$

Using (3.86), (3.88), and Fatou's Lemma,

$$\langle f, f \rangle_{H_0} \leq \liminf_{\epsilon \rightarrow 0} \langle f_\epsilon, f_\epsilon \rangle_{H_\epsilon} + \delta K = \Lambda + \delta K, \quad (3.90)$$

but since δ is arbitrary, this implies that

$$\langle f, f \rangle_{H_0} \leq \Lambda. \quad (3.91)$$

This combined with (3.89) and the uniqueness of eigenfunctions shows that $f = f_0$ is an eigenfunction for $\epsilon = 0$. This shows that the family $\{f_\epsilon\}$ converges weakly as $\epsilon \rightarrow 0$ to the eigenfunction f .

With these two preliminary results established, turn to the following expression:

$$\Gamma_\epsilon(f_\epsilon, f_\epsilon - f) = \Gamma_\epsilon(f_\epsilon, f_\epsilon) - \Gamma_\epsilon(f_\epsilon, f) \quad (3.92)$$

$$= \Lambda_\epsilon \int_0^\infty f_\epsilon^2 dx - \Lambda_\epsilon \int_0^\infty f_\epsilon f dx \quad (3.93)$$

$$= \Lambda_\epsilon - \Lambda_\epsilon \int_0^\infty f_\epsilon f dx. \quad (3.94)$$

This is well-defined since it has been shown that the variational characterization of eigen-pairs now holds for “test” functions taken from L^* , and $f_\epsilon \in L^*$ for all ϵ . This can also be expanded as follows.

$$\Gamma_\epsilon(f_\epsilon, f_\epsilon - f) = \Gamma_0(f_\epsilon, f_\epsilon - f) + \epsilon \int_0^\infty f_\epsilon^2 db - \epsilon \int_0^\infty f_\epsilon f db \quad (3.95)$$

$$= \Gamma_0(f_\epsilon, f_\epsilon) - \Gamma_0(f_\epsilon, f) + \epsilon \int_0^\infty f_\epsilon^2 db - \epsilon \int_0^\infty f_\epsilon f db \quad (3.96)$$

$$= \Gamma_\epsilon(f_\epsilon, f_\epsilon) - \Lambda \int_0^\infty f_\epsilon f dx - \epsilon \int_0^\infty f_\epsilon f db \quad (3.97)$$

$$= \Lambda_\epsilon - \Lambda \int_0^\infty f_\epsilon f dx - \epsilon \int_0^\infty f_\epsilon f db. \quad (3.98)$$

Equating these two expansions gives

$$\Lambda_\epsilon - \Lambda_\epsilon \int_0^\infty f_\epsilon f dx = \Lambda_\epsilon - \Lambda \int_0^\infty f_\epsilon f dx - \epsilon \int_0^\infty f_\epsilon f db \quad (3.99)$$

$$-\Lambda_\epsilon \int_0^\infty f_\epsilon f dx = -\Lambda \int_0^\infty f_\epsilon f dx - \epsilon \int_0^\infty f_\epsilon f db \quad (3.100)$$

$$(\Lambda_\epsilon - \Lambda) \int_0^\infty f_\epsilon f dx = \epsilon \int_0^\infty f_\epsilon f db. \quad (3.101)$$

Using the convergence established above,

$$\lim_{\epsilon \rightarrow 0} \frac{\Lambda_\epsilon - \Lambda}{\epsilon} = \int_0^\infty f^2 db. \quad (3.102)$$

This convergence is almost sure. This completes the proof.

□

Remark 9. Recall from Theorem 2.5.1 that the largest eigenvalue of the β -Hermite ensemble converges to the smallest eigenvalue of the Stochastic Airy Operator. That is why Λ_ϵ is paired with $f \equiv f_1(x) = Ai(x + a_1)/Ai'(a_1)$ in the limit above.

This result can be extended by considering the k th smallest eigenvalue, $\Lambda_{k-1,\epsilon} = \Lambda(k-1, \epsilon)$. In the above setting, the eigenvalue under consideration was $\Lambda_{0,\epsilon}$. Proceeding by the exact same argument, it can be shown that

$$\lim_{\epsilon \rightarrow 0} \frac{\Lambda(k-1, \epsilon) - \Lambda(k-1, 0)}{\epsilon} = \int_0^\infty f_k^2 db, \quad (3.103)$$

where

$$f_k(x) = \frac{Ai(x + a_k)}{Ai'(a_k)}. \quad (3.104)$$

The Brownian motion in all of the expressions is the same since it comes from the Brownian term in the bilinear form. Combining this with the a.s. convergence of the individual eigenvalues gives convergence in distribution for any finite collection of eigenvalues. Since it has been shown that the covariance structure is given by the same integrals that arise when $\beta \rightarrow \infty$ and then $n \rightarrow \infty$, this leads to the following corollary:

Corollary 3.3.2. *Let $\Lambda_{\epsilon,i}$ denote the $i+1$ th smallest eigenvalue of the Stochastic Airy Operator. Then, for any fixed k ,*

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\Lambda_{\epsilon,0} - \Lambda_{0,0}, \dots, \Lambda_{\epsilon,k-1} - \Lambda_{0,k-1}) = \tilde{G}, \quad (3.105)$$

jointly in law, where $\tilde{G} = (G_1, \dots, G_k)$ is a k -variate Gaussian vector with covariance structure given by (3.8). This is precisely the covariance structure found by taking $\beta \rightarrow \infty$ first and then letting $n \rightarrow \infty$.

3.4 Random Differential Operator

Given the eigenvalue problem $Lf = \lambda Mf$, there is a formal perturbation approach to determining the eigenvalues. Assume that L and M can be decomposed into deterministic and random parts as $L = L_0 + \alpha L_1$, $M = M_0 + \beta M_1$, where a 0 subscript indicates the deterministic part and the 1 subscript the random part, and the α and β are the perturbation parameters. Assume that

the following expansions are valid:

$$\lambda = \sum_{j,k=0}^{\infty} \lambda_{jk} \alpha^j \beta^k \quad (3.106)$$

$$f(x) = \sum_{j,k=0}^{\infty} f_{jk}(x) \alpha^j \beta^k. \quad (3.107)$$

Substituting the expansions into the original equation and equating coefficients gives the following system of equations:

$$L_0 f_{00} = \lambda_{00} M_0 f_{00} \quad (3.108)$$

$$L_0 f_{10} + L_1 f_{00} = \lambda_{00} M_0 f_{10} + \lambda_{10} M_0 f_{00} \quad (3.109)$$

$$\vdots \quad (3.110)$$

$$L_0 f_{rs} + L_1 f_{r-1,s} = \lambda_{rs} M_0 f_{00} + \sum_{j=0}^r \sum_{k=0}^{s-1} \lambda_{jk} (M_0 f_{r-j,s-k} + M_1 f_{r-j,s-k-1}) \quad (3.111)$$

The first equation is deterministic, and λ_{00} and f_{00} correspond to an eigenvalue/eigenvector pair for the deterministic equation. Let $\lambda_{00} := \mu$, $f_{00}(x) := \phi(x)$. Solving the next set of equations gives

$$\lambda_{10} = \frac{(L_1 \phi, \phi)}{(M_0 \phi, \phi)}, \quad \lambda_{01} = -\mu \frac{(M_1 \phi, \phi)}{(M_0 \phi, \phi)}, \quad (3.112)$$

where

$$(f, g) := \int_a^b f(x) g(x) dm(x), \quad (3.113)$$

denotes the inner-product on the Hilbert space $L^2[dm(x), [a, b]]$.

This then gives a perturbation expansion for λ :

$$\lambda = \mu + \frac{(L_1 \phi, \phi)}{(M_0 \phi, \phi)} \alpha - \mu \frac{(M_1 \phi, \phi)}{(M_0 \phi, \phi)} \beta + \dots, \quad (3.114)$$

where the dots indicate higher-order terms.

If ϕ is normalized with respect to the quadratic form defined by $H(u) := (M_0 u, u)$, that is, $(M_0 \phi, \phi) = 1$, then the above expansion reduces to

$$\lambda = \mu + (L_1 \phi, \phi) \alpha - \mu (M_1 \phi, \phi) \beta + \dots \quad (3.115)$$

Remark 10. *It must be noted that the material presented here constitutes a purely formal argument. Nothing can be “proved” using it, at least not in any rigorous sense. These kinds of methods often are helpful to get some feeling for what the correct answer should be, and then other means are needed to actually prove that the formal answer is actually correct.*

To derive the first-order behavior of the eigenvalues of the β -Hermite ensemble using this method, start with the formal random differential operator

$$-\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}}b' \quad (3.116)$$

where $\epsilon = 2/\sqrt{\beta}$.

Let

$$L_0 = -\frac{d^2}{dx^2} + x \quad (3.117)$$

and

$$L_1 = b'. \quad (3.118)$$

The eigenvalue problem for this operator is,

$$(L_0 + \epsilon L_1)f = \lambda f. \quad (3.119)$$

The solution to the deterministic equation $L_0 f = \lambda f$ is given by Airy functions shifted by Airy roots. In order to normalize the eigenfunction, we have to solve the following equation for c :

$$c^2 \int_0^\infty Ai^2(x + a_i) dx = 1, \quad (3.120)$$

and since

$$\int_0^\infty Ai^2(x + a_i) dx = (Ai'(a_i))^2, \quad (3.121)$$

it follows that

$$c = \frac{1}{Ai'(a_i)}. \quad (3.122)$$

Therefore, the i -th normalized eigenfunction is

$$f_i(x) = \frac{Ai(x + a_i)}{Ai'(a_i)}.$$

Using the theory outlined above,

$$\lambda_i = \mu + \epsilon \langle L_1 f, f \rangle \quad (3.123)$$

from which it follows that

$$\lambda_i - \mu = \epsilon \langle L_1 f, f \rangle \quad (3.124)$$

$$= \epsilon \int_0^\infty b' f^2(x) dx \quad (3.125)$$

$$= \epsilon \int_0^\infty f^2(x) db. \quad (3.126)$$

Then,

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_i - \mu}{\epsilon} = \int_0^\infty f^2(x) db, \quad (3.127)$$

which becomes

$$\lim_{\beta \rightarrow \infty} \frac{\sqrt{\beta}(\lambda_i - \mu)}{2} = \int_0^\infty f^2(x) db, \quad (3.128)$$

after the change of parameter $\epsilon = 2/\sqrt{\beta}$. This the same limit found in (3.81), up to notation.

Chapter 4

Large β : Laguerre Ensemble

4.1 Introduction

This chapter will follow the same outline as the previous one. Starting with the tridiagonal model of Dumitriu and Edelman for the (β, a) -Laguerre ensemble, LLN and CLT results for the eigenvalues of a fixed rank matrix of size $n \times n$ as $\beta \rightarrow \infty$ will be derived. This is based on the same paper of Dumitriu and Edelman that was the main source for the results in the previous chapter. Unfortunately, for the (β, a) -Laguerre ensemble, these results have a number of errors. As the interested reader can find all of the original results in [DE05], this paper will not reproduce them in any detail. What will be done is to present the correct form of the results with detailed proof. Once the correct finite rank covariance matrix has been derived, the continuum limit will be found in terms of integrals of Bessel functions. The asymptotic behavior of the covariance matrix will then be investigated. Finally, the question as to whether the β and n limits commute will be explored.

4.2 (β, a) -Laguerre with n fixed and $\beta \rightarrow \infty$.

In what follows, $a > -1$ is fixed, and $l_1^{(n)}, \dots, l_n^{(n)}$ are the roots of the n th Laguerre polynomial of parameter a , which is denoted L_n^a . The matrices, B_β , are drawn from the (β, a) -Laguerre ensemble, of size $n \times n$, and scaled by $1/n\beta$.

Consider the following $n \times n$ matrices:

$$L_a = \begin{pmatrix} n+a & \sqrt{n+a}\sqrt{n-1} & & & & \\ \sqrt{n+a}\sqrt{n-1} & 2(n-1)+a & \sqrt{n+a-1}\sqrt{n-2} & & & \\ & \sqrt{n+a-1}\sqrt{n-2} & 2(n-2)+a & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \sqrt{3+a}\sqrt{2} & \\ & & & \sqrt{3+a}\sqrt{2} & 4+a & \sqrt{2+a}\sqrt{1} \\ & & & & \sqrt{2+a}\sqrt{1} & 2+a \end{pmatrix}$$

and

$$B_a = \begin{pmatrix} \sqrt{n+a} & & & & \\ \sqrt{n-1} & \sqrt{n+a-1} & & & \\ & \ddots & \ddots & & \\ & & \sqrt{2} & \sqrt{a+2} & \\ & & & \sqrt{1} & \sqrt{a+1} \end{pmatrix}$$

Note that $L_a = B_a B_a^T$.

The eigenvalues of L_a are the n roots of $L_n^a(x)$. For notational clarity, these will be denoted as l_i , $1 \leq i \leq n$, the dependence on n and a being omitted. The corresponding eigenvector is

$$w_i = \begin{pmatrix} L_{n-1}^{a+1}(l_i) \\ -L_{n-2}^{a+1}(l_i) \\ \vdots \\ (-1)^n L_1^{a+1}(l_i) \\ (-1)^{n+1} L_0^{a+1}(l_i) \end{pmatrix}$$

Using Lemmas 2.7.1 and 2.7.2 leads to,

$$\lambda_i(B_\beta) = \lambda_i \left(\frac{1}{n} L_a + \frac{1}{n\sqrt{2\beta}} (B_a Z^T + Z B_a^T) \right) + o \left(\frac{1}{\sqrt{\beta}} \right) \quad (4.1)$$

$$\lambda_i(B_\beta) = \frac{1}{n}l_i + \frac{1}{n\sqrt{2\beta}} \frac{w_i^T(B_a Z^T + Z B_a^T)w_i}{w_i^T w_i} + o\left(\frac{1}{\sqrt{\beta}}\right) \quad (4.2)$$

where

$$Z = \begin{pmatrix} M_n & & & & \\ N_{n-1} & M_{n-1} & & & \\ & \ddots & \ddots & & \\ & & N_2 & M_2 & \\ & & & N_1 & M_1 \end{pmatrix} \quad (4.3)$$

with all M_i and N_j independent standard Gaussian random variables.

From (4.2), the following result is immediate.

Lemma 4.2.1. (part of Theorem 4.1 of [DE05]) *Let $\lambda_i(B_\beta)$ be the i th largest eigenvalue of B_β , for any fixed $1 \leq i \leq n$. Then, as $\beta \rightarrow \infty$,*

$$\lambda_i(B_\beta) \rightarrow \frac{1}{n}l_i. \quad (4.4)$$

This result will be needed below.

Returning to the derivation of the covariance matrix, since $w_i^T B_a Z^T w_i = w_i^T Z B_a^T w_i$, (4.2) can be rewritten as

$$\lambda_i(B_\beta) = \frac{1}{n}l_i + \frac{\sqrt{2}}{n\sqrt{\beta}} \frac{w_i^T B_a Z^T w_i}{w_i^T w_i} + o\left(\frac{1}{\sqrt{\beta}}\right) \quad (4.5)$$

A computation gives

$$w_i^T B_a Z^T w_i = \sqrt{a+1}(L_0^{a+1}(l_i))^2 M_1 + \sum_{m=1}^{n-1} ((\sqrt{a+1+m}L_m^{a+1}(l_i) - \sqrt{m}L_{m-1}^{a+1}(l_i))L_m^{a+1}(l_i))M_{m+1} \quad (4.6)$$

$$+ \sum_{m=1}^{n-1} ((-\sqrt{a+1+m}L_m^{a+1}(l_i) + \sqrt{m}L_{m-1}^{a+1}(l_i))L_{m-1}^{a+1}(l_i))N_m. \quad (4.7)$$

Another computation gives

$$w_i^T w_i = \sum_{m=0}^{n-1} (L_m^{a+1}(l_i))^2. \quad (4.8)$$

From this, it follows that

$$\sqrt{\beta} \left(\lambda_i(B_\beta) - \frac{1}{n} l_i \right) = \frac{\sqrt{2} \sqrt{a+1} (L_0^{a+1}(l_i))^2 M_1 + \text{Sum}_1 + \text{Sum}_2}{\sum_{m=0}^{n-1} (L_m^{a+1}(l_i))^2} \quad (4.9)$$

where

$$\text{Sum}_1 = \sum_{m=1}^{n-1} ((\sqrt{a+1+m} L_m^{a+1}(l_i) - \sqrt{m} L_{m-1}^{a+1}(l_i)) L_m^{a+1}(l_i)) M_{m+1} \quad (4.10)$$

$$\text{Sum}_2 = \sum_{m=1}^{n-1} ((-\sqrt{a+1+m} L_m^{a+1}(l_i) + \sqrt{m} L_{m-1}^{a+1}(l_i)) L_{m-1}^{a+1}(l_i)) N_m \quad (4.11)$$

Let

$$\tilde{\lambda}_i \equiv \sqrt{\beta} \left(\lambda_i(B_\beta) - \frac{1}{n} l_i \right). \quad (4.12)$$

Then each $\tilde{\lambda}_i$ is a linear combination of the $2n-1$ i.i.d. standard Gaussian random variables $\{M_i\}_{i=1}^n$ and $\{N_i\}_{i=1}^{n-1}$. Let $\Lambda \equiv (\tilde{\lambda}_1, \dots, \tilde{\lambda}_n)$ and $\mathbf{M} = (M_1, \dots, M_n, N_1, \dots, N_{n-1})$. From the theory of multivariate Gaussian random variables,

$$\Lambda = A\mathbf{M} \quad (4.13)$$

where the $n \times (2n-1)$ matrix A has entries given by

$$a_{im} = \begin{cases} \frac{(\sqrt{a+1+m-1} L_{m-1}^{a+1}(l_i) - \sqrt{m-1} L_{m-2}^{a+1}(l_i)) L_{m-1}^{a+1}(l_i)}{\sum_{n=0}^{n-1} (L_n^{a+1}(l_i))^2} & \text{if } 1 \leq m \leq n \\ \frac{(\sqrt{a+1+m-n} L_{m-n}^{a+1}(l_i) - \sqrt{m-n} L_{m-n-1}^{a+1}(l_i)) L_{m-n-1}^{a+1}(l_i)}{\sum_{n=0}^{n-1} (L_n^{a+1}(l_i))^2} & \text{if } n+1 \leq m \leq 2n-1 \end{cases}$$

This means that the n -variate random vector Λ is Gaussian with mean zero and covariance matrix $\Sigma = AA^T$.

Use the identity

$$L_n^{a+1}(x) = \sqrt{n+a+2} L_n^{a+2}(x) - \sqrt{n} L_{n-1}^{a+2}(x) \quad (4.14)$$

to rewrite

$$-\sqrt{n} L_{n-1}^{a+1}(l_i) = L_n^a(l_i) - \sqrt{n+a+1} L_n^{a+1}(l_i). \quad (4.15)$$

Then the covariance matrix simplifies to

$$\Sigma_{ij} = \frac{1}{\left(\sum_{m=0}^{n-1} (L_m^{a+1}(l_i))^2\right) \left(\sum_{m=0}^{n-1} (L_m^{a+1}(l_j))^2\right)} \left((a+1)(L_0^{a+1}(l_i))^2 (L_0^{a+1}(l_j))^2 \right. \\ \left. + \sum_{m=1}^{n-1} L_m^a(l_i) L_m^a(l_j) L_m^{a+1}(l_i) L_m^{a+1}(l_j) + \sum_{m=1}^{n-1} L_m^a(l_i) L_m^a(l_j) L_{m-1}^{a+1}(l_i) L_{m-1}^{a+1}(l_j) \right).$$

This completes the proof of the following

Theorem 4.2.2. *Let $\lambda_i(B_\beta)$ be the i th smallest eigenvalue of B_β , for any fixed $1 \leq i \leq n$. Then*

$$\lim_{\beta \rightarrow \infty} \sqrt{\beta} \left(\lambda_1(B_\beta) - \frac{1}{n} l_1, \lambda_2(B_\beta) - \frac{1}{n} l_2, \dots, \lambda_n(B_\beta) - \frac{1}{n} l_n \right) = \frac{\sqrt{2}}{n} G \quad (4.16)$$

where $G \equiv (G_1, G_2, \dots, G_n)$ is a centered n -variate Gaussian random vector with covariance matrix

$$\Sigma_{ij} = \frac{1}{\left(\sum_{m=0}^{n-1} (L_m^{a+1}(l_i))^2\right) \left(\sum_{m=0}^{n-1} (L_m^{a+1}(l_j))^2\right)} \left((a+1)(L_0^{a+1}(l_i))^2 (L_0^{a+1}(l_j))^2 \right. \\ \left. + \sum_{m=1}^{n-1} L_m^a(l_i) L_m^a(l_j) L_m^{a+1}(l_i) L_m^{a+1}(l_j) + \sum_{m=1}^{n-1} L_m^a(l_i) L_m^a(l_j) L_{m-1}^{a+1}(l_i) L_{m-1}^{a+1}(l_j) \right). \quad (4.17)$$

The convergence here is of p.d.f.'s, uniformly on any fixed product of intervals.

Remark 11. While the proof of this theorem presented above follows almost exactly the proof in [DE05], it must be pointed out that the results presented in that paper are incorrect. This is due to the fact that they did not use the correct eigenvector when they used Lemma 2.7.1 on page 19. The eigenvector used in their paper is

$$\tilde{w}_i = \begin{pmatrix} L_{n-1}^{a+1}(l_i) \\ L_{n-2}^{a+1}(l_i) \\ \vdots \\ L_1^{a+1}(l_i) \\ L_0^{a+1}(l_i) \end{pmatrix}, \text{ not } w_i = \begin{pmatrix} L_{n-1}^{a+1}(l_i) \\ -L_{n-2}^{a+1}(l_i) \\ \vdots \\ (-1)^n L_1^{a+1}(l_i) \\ (-1)^{n+1} L_0^{a+1}(l_i) \end{pmatrix} \quad (4.18)$$

This same simplification was made in the β -Hermite case as well, but there it caused no harm, because the minus sign could be absorbed into the Gaussian random variables. In this case, the minus signs affects the coefficients of the Gaussians, which aren't single orthogonal polynomials as

they were in the Hermite case, but instead the difference of Laguerre polynomials. This can't be absorbed into the Gaussian, because now the absolute value of the coefficients has been changed.

Remark 12. The order of matrix multiplication, BB^T or B^TB , should have no effect on the spectrum of the resulting tridiagonal matrix. Indeed, this was one of the original indications that the work in [DE05] was not quite right. A few words will be said here about this situation.

Start with the given eigenvalue problem: $BB^Tw_i = \lambda w_i$. Multiplying on the left by B^T changes it to $B^TB(B^Tw_i) = \lambda(B^Tw_i)$. Then the eigenvector for B^TB is given by $v_i = B^Tw_i$, which can be simplified using Laguerre relations to

$$v_i = \begin{pmatrix} L_{n-1}^a(l_{i,n}^a) \\ -L_{n-2}^a(l_{i,n}^a) \\ \vdots \\ (-1)^n L_1^a(l_{i,n}^a) \\ (-1)^{n+1} L_0^a(l_{i,n}^a) \end{pmatrix}$$

The eigenvalues are the same, and this eigenvector has the nice property that the order of the Laguerre polynomials is the same as the order of the roots.

Because of the relationship between w_i and v_i , (4.2) can also be written as

$$\begin{aligned} \lambda_i(B_\beta) &\sim \frac{1}{n} l_{i,k}^a + \frac{1}{n\sqrt{2\beta}} \frac{v_i^T B^{-1} (BZ^T + ZB^T) (B^{-1})^T v_i}{v_i (B^T B)^{-1} v_i} \\ &\sim \frac{1}{n} l_{i,k}^a + \frac{1}{n\sqrt{2\beta}} \frac{v_i^T (Z^T (B^{-1})^T + B^{-1} Z) v_i}{v_i (B^T B)^{-1} v_i} \end{aligned}$$

and this can be analyzed in the manner described above. It should be noted that B^{-1} is a lower triangular matrix, so the resulting computations become much more involved.

4.3 (β, a) -Laguerre for $n \rightarrow \infty$ and $\beta \rightarrow \infty$

The main results of this chapter are:

Theorem 4.3.1. (Law of Large Numbers) *Let λ_i be the i th smallest eigenvalue of the $n \times n$ matrix B_β and let $j_{i,a}$ be the i th positive root of $J_a(x)$. Then*

$$\lim_{n \rightarrow \infty} \lim_{\beta \rightarrow \infty} n^2(\lambda_i(B_\beta)) = \frac{j_{i,a}^2}{4}. \quad (4.19)$$

Proof. From Lemma 4.2.1,

$$\lim_{\beta \rightarrow \infty} \lambda_i(B_\beta) = \frac{1}{n} l_i. \quad (4.20)$$

Using the asymptotic formula for the roots of Laguerre polynomials (see [Sze03]),

$$\lim_{n \rightarrow \infty} n l_i = \frac{j_{i,a}^2}{4}. \quad (4.21)$$

Combining these two limits completes the proof. □

In the case of the smallest eigenvalue, the above theorem shows that it converges to the first positive root of the Bessel function of order a . As $n \rightarrow \infty$, this root is converging to zero while always staying positive. This illustrates the hard constraint at the origin.

Theorem 4.3.2. (Central Limit Theorem) *Let λ_i be the i th smallest eigenvalue of the $n \times n$ matrix B_β and let $j_{i,a}$ be the i th positive root of $J_a(x)$. Then,*

$$\lim_{n \rightarrow \infty} \lim_{\beta \rightarrow \infty} n^2 \sqrt{\beta} \left(\lambda_1(B_\beta) - \frac{1}{n} l_1, \lambda_2(B_\beta) - \frac{1}{n} l_2, \dots, \lambda_n(B_\beta) - \frac{1}{n} l_n \right) = \tilde{G}, \quad (4.22)$$

where $\tilde{G} \equiv (\tilde{G}_1, \dots, \tilde{G}_n, \dots)$ is an infinite Gaussian random vector with covariance matrix $\tilde{\Sigma}$, where

$$\tilde{\Sigma}_{ik} = \text{Cov}(\tilde{G}_i, \tilde{G}_k) = \frac{j_{i,a} j_{k,a} \int_0^1 J_a(j_{i,a} \sqrt{x}) J_a(j_{k,a} \sqrt{x}) J_{a+1}(j_{i,a} \sqrt{x}) J_{a+1}(j_{k,a} \sqrt{x}) dx}{\left(\int_0^1 J_{a+1}^2(j_{i,a} \sqrt{x}) dx \right) \left(\int_0^1 J_{a+1}^2(j_{k,a} \sqrt{x}) dx \right)} \quad (4.23)$$

As in the β -Hermite case, the proof of (4.22) follows from the proof of (4.23) and the finite n result, Theorem 4.2.2. Since the details are exactly analogous to the previous case, they will not be repeated here.

To prove (4.23), start with the covariance matrix for a fixed n :

$$\Sigma_{ij} = \frac{1}{\left(\sum_{m=0}^{n-1} (L_m^{a+1}(l_i))^2 \right) \left(\sum_{m=0}^{n-1} (L_m^{a+1}(l_j))^2 \right)} ((a+1)(L_0^{a+1}(l_i))^2 (L_0^{a+1}(l_j))^2$$

$$+ \sum_{m=1}^{n-1} L_m^a(l_i) L_m^a(l_j) L_m^{a+1}(l_i) L_m^{a+1}(l_j) + \sum_{m=1}^{n-1} L_m^a(l_i) L_m^a(l_j) L_{m-1}^{a+1}(l_i) L_{m-1}^{a+1}(l_j) \Bigg).$$

It must be shown that these sums of Laguerre polynomials converge, with the appropriate scaling, to the integrals of Bessel functions shown in Theorem 4.3.2. As this proof follows very closely that of Theorem 3.2.2, only the main steps will be given below.

Proof. Look at the first sum in the numerator in the situation that $i = j$. The following asymptotic formula of Hilb's type (see [Sze03]) relates Laguerre polynomials and Bessel functions:

$$e^{-x/2} x^{a/2} \tilde{L}_n^{(a)}(x) = N^{-a/2} \frac{\Gamma(n+a+1)}{n!} J_a(2\sqrt{Nx}) + \epsilon_n, \quad (4.24)$$

where $a > -1$, $N = \frac{2n+a+1}{2}$, and $x > 0$. Using this, the sum becomes

$$\begin{aligned} & \sum_{m=1}^{n-1} (L_m^a(l_i) L_m^{a+1}(l_i))^2 \\ & \approx e^{2l_i} l_i^{-(2a+1)} 2^{2a+1} \end{aligned} \quad (4.25)$$

$$\times \sum (2m+a+1)^{-a} \left(\frac{\Gamma(m+a+1)}{\Gamma(m+1)} \right) (2m+a+2)^{-(a+1)} \left(\frac{\Gamma(m+a+2)}{\Gamma(m+1)} \right) J_a^2(\tilde{x}) J_{a+1}^2(\tilde{x}) \quad (4.26)$$

$$\approx e^{2l_i} l_i^{-(2a+1)} 2^{2a+1} \sum (2m)^{-a} m^a (2m)^{-(a+1)} m^{a+1} J_a^2(\tilde{x}) J_{a+1}^2(\tilde{x}) \quad (4.27)$$

$$= e^{2l_i} l_i^{-(2a+1)} \sum J_a^2(j_i \left(\frac{m}{n}\right)^{1/2}) J_{a+1}^2(j_i \left(\frac{m}{n}\right)^{1/2}) \quad (4.28)$$

$$= e^{2l_i} l_i^{-(2a+1)} n \sum \frac{1}{n} J_a^2\left(j_i \left(\frac{m}{n}\right)^{1/2}\right) J_{a+1}^2\left(j_i \left(\frac{m}{n}\right)^{1/2}\right) \quad (4.29)$$

$$= e^{2l_i} l_i^{-(2a+1)} n \sum J_a^2(j_i \tilde{x}_m^{1/2}) J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x \quad (4.30)$$

Throughout, $\tilde{x} = 2\sqrt{Nl_i}$ with $N = (2m+a)/2$. This is analyzed as follows:

$$2\sqrt{Nl_i} = 2 \left(\frac{(2m+a)l_i}{2} \right)^{1/2} \quad (4.31)$$

$$\approx 2 \left(\frac{(2m)(j_i^2/4n)}{2} \right)^{1/2} \quad (4.32)$$

$$= 2 \left(\frac{j_i^2 m}{4n} \right)^{1/2} \quad (4.33)$$

$$= j_i \sqrt{m/n}. \quad (4.34)$$

This gives $\Delta x = 1/n$ as the step size for the Riemann sum.

In going from (4.26) to (4.27), the asymptotic relation

$$\lim_{n \rightarrow \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1, \quad (4.35)$$

was used. This standard fact can be found in [OLBC10].

For the denominator,

$$\left(\sum_{m=0}^{n-1} (L_m^{a+1}(l_i))^2 \right)^2 \approx e^{2l_i} l_i^{-2(a+1)} n^2 \left(\sum_{m=0}^{n-1} \frac{1}{n} J_{a+1}^2 \left(j_i \left(\frac{m}{n} \right)^{1/2} \right) \right)^2 \quad (4.36)$$

$$= e^{2l_i} l_i^{-2(a+1)} n^2 \left(\sum_{m=0}^{n-1} J_{a+1}^2 \left(j_i \tilde{x}_m^{1/2} \right) \Delta x \right)^2 \quad (4.37)$$

Taking the ratio of these two Riemann sums gives

$$\frac{e^{2l_i} l_i^{-(2a+1)} n \sum J_a^2(j_i \tilde{x}_m^{1/2}) J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x}{e^{2l_i} l_i^{-2(a+1)} n^2 \left(\sum J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x \right)^2} = \frac{l_i \sum J_a^2(j_i \tilde{x}_m^{1/2}) J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x}{n \left(\sum J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x \right)^2} \quad (4.38)$$

Finally, consider the term in the numerator in front of the sums,

$$\frac{(a+1)(L_0^{a+1}(l_i))^2 (L_0^{a+1}(l_j))^2}{\left(\sum_{m=1}^{n-1} (L_m^{a+1}(l_i))^2 \right) \left(\sum_{m=1}^{n-1} (L_m^{a+1}(l_j))^2 \right)}. \quad (4.39)$$

The numerator is a constant that doesn't depend on n , whereas the denominator goes to ∞ as $n \rightarrow \infty$. This means that this term goes to 0 in the limit and doesn't contribute anything to the resulting integrals.

Recall that we are trying to prove

$$\lim_{n \rightarrow \infty} \lim_{\beta \rightarrow \infty} n^2 \sqrt{\beta} \left(\lambda_1(B_\beta) - \frac{1}{n} l_1, \lambda_2(B_\beta) - \frac{1}{n} l_2, \dots, \lambda_n(B_\beta) - \frac{1}{n} l_n \right) = \tilde{G}, \quad (4.40)$$

From Theorem 4.2.2, the β limit is

$$\lim_{\beta \rightarrow \infty} \sqrt{\beta} \left(\lambda_1(B_\beta) - \frac{1}{n} l_1, \lambda_2(B_\beta) - \frac{1}{n} l_2, \dots, \lambda_n(B_\beta) - \frac{1}{n} l_n \right) = \frac{\sqrt{2}}{n} G \quad (4.41)$$

Combining these two gives

$$\lim_{n \rightarrow \infty} n^2 \lim_{\beta \rightarrow \infty} \sqrt{\beta} \left(\lambda_1(B_\beta) - \frac{1}{n} l_1, \dots, \lambda_n(B_\beta) - \frac{1}{n} l_n \right) = \lim_{n \rightarrow \infty} \frac{\sqrt{2} n^2}{n} G = \lim_{\sqrt{2} n \rightarrow \infty} n G \quad (4.42)$$

Let $\tilde{G}^{(n)} \equiv (\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n) = nG \equiv (nG_1, nG_2, \dots, nG_n)$ be the first n terms of \tilde{G} . From the Riemann sum computation above, the variance for $\tilde{G}_i^{(n)}$ is given by

$$\frac{n^2 l_i \sum J_a^2(j_i \tilde{x}_m^{1/2}) J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x}{n \left(\sum J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x \right)^2} = \frac{n l_i \sum J_a^2(j_i \tilde{x}_m^{1/2}) J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x}{\left(\sum J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x \right)^2} \quad (4.43)$$

Using (4.21) for l_i as $n \rightarrow \infty$ leads to

$$\lim_{n \rightarrow \infty} \frac{n l_i \sum J_a^2(j_i \tilde{x}_m^{1/2}) J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x}{\left(\sum J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x \right)^2} = \lim_{n \rightarrow \infty} \frac{n \frac{j_{i,a}^2}{4n} \sum J_a^2(j_i \tilde{x}_m^{1/2}) J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x}{\left(\sum J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x \right)^2} \quad (4.44)$$

$$= \lim_{n \rightarrow \infty} \frac{j_{i,a}^2 \sum J_a^2(j_i \tilde{x}_m^{1/2}) J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x}{4 \left(\sum J_{a+1}^2(j_i \tilde{x}_m^{1/2}) \Delta x \right)^2} \quad (4.45)$$

$$= \frac{j_{i,a}^2 \int_0^1 J_{a+1}^2(j_{i,a} \sqrt{x}) J_a^2(j_{i,a} \sqrt{x}) dx}{4 \left(\int_0^1 J_{a+1}^2(j_{i,a} \sqrt{x}) dx \right)^2} \quad (4.46)$$

By an argument analogous to the Hermite case, the second sum will also converge to the expression given above. This gives a 2 in the numerator. The $\sqrt{2}$ in front of G becomes a 2 in the variance, so together this becomes a 4. This cancels with the 4 in the denominator that came from the asymptotics for l_i . This gives

$$\text{Var}(\tilde{G}_i) = \frac{j_{i,a}^2 \int_0^1 J_a^2(j_i x^{1/2}) J_{a+1}^2(j_i x^{1/2}) dx}{\left(\int_0^1 J_{a+1}^2(j_i x^{1/2}) dx \right)^2} \quad (4.47)$$

Finally, when $i \neq j$, all that changes is the indices for half of the Laguerre roots. All of the asymptotic results hold, as does everything involving the Riemann sums. This completes the proof. \square

Remark 13. *The asymptotics of the covariance matrix are not as amenable to analysis as in the β -Hermite case. The main difficulty seems to stem from the fact that, when $i \neq j$, the integrand in the numerator takes both positive and negative values, and the cancelation in the resulting integral can't be readily quantified. From the numerical evidence, it appears that the covariance decrease across rows to a finite, non-zero limit. The variance, on the other hand, grows rapidly as i gets large. The cause of this seems to be the $j_{i,a}^2$ term in front of the integrals.*

4.4 Commuting the limits.

For this section, define $\epsilon = 2/\sqrt{\beta}$, and let

$$K_\epsilon(x, y) = \int_0^{x \wedge y} e^{az + \epsilon b(z)} dz. \quad (4.48)$$

Then $K_\epsilon(x, y)$ is the kernel of the operator

$$(\mathfrak{G}_{\beta, a}^{-1} \psi)(x) = \int_0^\infty \left(\int_0^{x \wedge y} e^{az + \epsilon b(z)} dz \right) \psi(y) e^{-(a+1)y - \epsilon b(y)} dy. \quad (4.49)$$

It can be shown (see [RR09]) that $\mathfrak{G}_{\beta, a}^{-1}$ is non-negative symmetric on $L^2[\mathbb{R}_+, e^{-(a+1)x - \epsilon b(x)}]$. To remove the dependence of the L^2 space on the parameters a and ϵ , symmetrize the operator to get the following

$$(\tilde{\mathfrak{G}}_{\beta, a}^{-1} \psi)(x) = \int_0^\infty \left(\int_0^{x \wedge y} e^{az + \epsilon b(z)} dz \right) \psi(y) e^{-(a+1)/2(x+y) - \epsilon/2(b(x) + b(y))} dy, \quad (4.50)$$

which is now defined on $L^2[\mathbb{R}_+]$ for any value of a and β . For notational clarity, the β and a subscripts will be replaced by a single subscript for ϵ , and the operator $\tilde{\mathfrak{G}}_\epsilon^{-1}$ will be denoted by H_ϵ . The eigenfunctions and eigenvectors corresponding to H_ϵ will be denoted by f_ϵ and λ_ϵ .

For the rest of the section, assume that it is only the ground-state eigenvalue and corresponding eigenfunction that is being considered. This means that $\lambda_0 = j_{a,1}$, the smallest root of the Bessel function of order a , which will be denoted throughout by j_a . It can be shown, either by differentiation or direct computation, that the deterministic eigenfunctions of H_0 are given by

$$f_0(x) = ce^{-(a+1)/2x} e^{ax/2} J_a(j_a e^{-x/2}) \quad (4.51)$$

$$= ce^{-x/2} J_a(j_a e^{-x/2}). \quad (4.52)$$

In the above, c is a normalization constant.

Since H_ϵ is a compact operator mapping $L^2 \rightarrow L^2$,

$$\langle f, H_\epsilon f_\epsilon \rangle = \lambda_\epsilon \langle f, f_\epsilon \rangle \quad (4.53)$$

for any $f \in L^2$. In particular, since $f_0 \in L^2$, it follows that

$$\lambda_\epsilon \langle f_0, f_\epsilon \rangle = \langle f_0, H_\epsilon f_\epsilon \rangle. \quad (4.54)$$

Starting with this expression and following the technique used in the β -Hermite case gives

$$\lambda_\epsilon \langle f_0, f_\epsilon \rangle = \langle f_0, H_\epsilon f_\epsilon \rangle \quad (4.55)$$

$$= \langle f_0, (H_\epsilon - H_0) f_\epsilon \rangle + \langle f_0, H_0 f_\epsilon \rangle \quad (4.56)$$

$$= \langle f_0, (H_\epsilon - H_0) f_\epsilon \rangle + \langle H_0 f_0, f_\epsilon \rangle \quad (4.57)$$

$$= \langle f_0, (H_\epsilon - H_0) f_\epsilon \rangle + \lambda \langle f_0, f_\epsilon \rangle. \quad (4.58)$$

The fact that H_0 is a symmetric operator was used in (4.57). Rearranging the above leads to

$$(\lambda_\epsilon - \lambda) \langle f_0, f_\epsilon \rangle = \langle f_\epsilon, (H_\epsilon - H_0) f_0 \rangle \quad (4.59)$$

$$\frac{1}{\epsilon} (\lambda_\epsilon - \lambda) \langle f_0, f_\epsilon \rangle = \langle f_\epsilon, \frac{H_\epsilon - H_0}{\epsilon} f_0 \rangle. \quad (4.60)$$

Consider just the difference of operators.

$$H_\epsilon - H_0 = \int_0^\infty K_\epsilon(x, y) f_0(y) e^{-\frac{a+1}{2}(x+y) - \frac{\epsilon}{2}(b(x)+b(y))} dy - \int_0^\infty K_0(x, y) f_0(y) e^{-\frac{a+1}{2}(x+y)} dy \quad (4.61)$$

$$= \int_0^\infty K_\epsilon(x, y) f_0(y) e^{-\frac{a+1}{2}(x+y) - \frac{\epsilon}{2}(b(x)+b(y))} dy - \int_0^\infty K_0(x, y) f_0(y) e^{-\frac{a+1}{2}(x+y)} dy \quad (4.62)$$

$$\begin{aligned} & + \int_0^\infty K_0(x, y) f_0(y) e^{-\frac{a+1}{2}(x+y)} e^{-\frac{\epsilon}{2}(b(x)+b(y))} dy \\ & - \int_0^\infty K_0(x, y) f_0(y) e^{-\frac{a+1}{2}(x+y)} e^{-\frac{\epsilon}{2}(b(x)+b(y))} dy \\ & = \int_0^\infty (K_\epsilon(x, y) - K_0(x, y)) f_0(y) e^{-\frac{a+1}{2}(x+y)} e^{-\frac{\epsilon}{2}(b(x)+b(y))} dt \\ & - \int_0^\infty K_0(x, y) f_0(y) e^{-\frac{a+1}{2}(x+y)} \left(e^{-\frac{\epsilon}{2}(b(x)+b(y))} - 1 \right) dy. \end{aligned} \quad (4.63)$$

As $\epsilon \rightarrow 0$,

$$\frac{e^{-\frac{\epsilon}{2}(b(x)+b(y))} - 1}{\epsilon} \Rightarrow -\frac{b(x) + b(y)}{2}, \quad (4.64)$$

$$\frac{K_\epsilon(x, y) - K_0(x, y)}{\epsilon} = \int_0^{x \wedge y} e^{az} \left(\frac{e^{\epsilon b(z)} - 1}{\epsilon} \right) dz \Rightarrow \int_0^{x \wedge y} e^{az} b(z) dz. \quad (4.65)$$

Putting this back into (4.60) and taking $\epsilon \rightarrow 0$ gives

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon - \lambda_0}{\epsilon} &= \int_0^\infty \int_0^\infty \left(\int_0^{x \wedge y} e^{az} b(z) dz \right) f_0(x) f_0(y) e^{-\frac{a+1}{2}(x+y)} dy dx \\ &+ \frac{1}{2} \int_0^\infty \int_0^\infty \left(\int_0^{x \wedge y} e^{az} dz \right) f_0(x) f_0(y) e^{-\frac{a+1}{2}(x+y)} (b(x) + b(y)) dy dx \end{aligned} \quad (4.66)$$

The integrals on the right-hand side are averages over Brownian paths, so they are mean-zero Gaussian. To simplify the next computation, let

$$G_1(x, y) = \left(\int_0^{x \wedge y} e^{az} b(z) dz \right) e^{-\frac{a+1}{2}(x+y)} \quad (4.67)$$

$$G_2(x, y) = \left(\int_0^{x \wedge y} e^{az} dz \right) e^{-\frac{a+1}{2}(x+y)} (b(x) + b(y)). \quad (4.68)$$

Then the variance is computed as

$$E \left[\lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon - \lambda_0}{\epsilon} \right]^2 = E \left[\int_0^\infty \int_0^\infty G_1(x, y) f_0(x) f_0(y) dx dy + \int_0^\infty \int_0^\infty G_2(x, y) f_0(x) f_0(y) dx dy \right]^2 \quad (4.69)$$

$$= E \left[\int_0^\infty \int_0^\infty G_1(x, y) f_0(x) f_0(y) dx dy \right]^2 + E \left[\int_0^\infty \int_0^\infty G_2(x, y) f_0(x) f_0(y) dx dy \right]^2 \quad (4.70)$$

$$+ 2E \left[\int_0^\infty \int_0^\infty G_1(x, y) f_0(x) f_0(y) dx dy \int_0^\infty \int_0^\infty G_2(x, y) f_0(x) f_0(y) dx dy \right] \quad (4.71)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E[G_1(x, y) G_1(x', y')] f_0(x) f_0(y) f_0(x') f_0(y') dx dy dx' dy' \quad (4.72)$$

$$+ \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E[G_2(x, y) G_2(x', y')] f_0(x) f_0(y) f_0(x') f_0(y') dx dy dx' dy' \quad (4.73)$$

$$+ 2 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty E[G_1(x, y) G_2(x', y')] f_0(x) f_0(y) f_0(x') f_0(y') dx dy dx' dy'. \quad (4.74)$$

To get some idea of what these integrals look like, start with the $E[G_2(x, y) G_2(x', y')]$ term given by (4.73). Using the fact that $E[b(x)b(y)] = x \wedge y$ and the symmetry of the integrand, this becomes

$$\begin{aligned} & \frac{1}{a^2} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{a+2}{2}(x+y+x'+y')} J_a(j_a e^{-x/2}) J_a(j_a e^{-y/2}) J_a(j_a e^{-x'/2}) J_a(j_a e^{-y'/2}) \\ & \quad \times (e^{a(x \wedge y)} - 1)(e^{a(x' \wedge y')} - 1)(x \wedge x') dx dy dx' dy'. \end{aligned}$$

There are similar expressions for the other two expectations. Due to the min's throughout the integrals, explicit evaluation becomes very tricky. By exploiting the highly symmetric nature of the

integrands, some simplification is possible. For example, here is (4.72):

$$\begin{aligned} & \frac{4}{a^2} \int_0^\infty \int_0^x \int_0^\infty \int_0^{x'} e^{a(y+y')} e^{-(a+1)(x+y+x'+y')} (y \wedge y') \\ & \quad \times J_a(j_a e^{-x/2}) J_a(j_a e^{-y/2}) J_a(j_a e^{-x'/2}) J_a(j_a e^{-y'/2}) dy' dx' dy dx \\ & - \frac{4}{2a^3} \int_0^\infty \int_0^x \int_0^\infty \int_0^{x'} e^{-(a+1)(x+y+x'+y')} (e^{2a(y \wedge y')} - 1) \\ & \quad \times J_a(j_a e^{-x/2}) J_a(j_a e^{-y/2}) J_a(j_a e^{-x'/2}) J_a(j_a e^{-y'/2}) dy' dx' dy dx. \end{aligned}$$

As a final simplification, make the change of variable $u = e^{-x}$, done for all of the variables. This leads to integrals on $[0, 1]$ of Bessel functions of the form $J_a(j_a \sqrt{x})$. This change results in integrals that are tantalizingly close to those computed in Theorem 4.3.2.

From these considerations,

$$\frac{\lambda_\epsilon - \lambda}{\epsilon} \sim N(0, \Gamma)$$

where Γ is found through explicit calculation of the above integrals. Since these are eigenvalues of the integral operator, they are the reciprocals of the eigenvalues of the differential operators in the Dumitriu and Edelman paper. Let Λ_ϵ be the corresponding eigenvalue of the differential operator, so that

$$\lambda_\epsilon = \frac{1}{\Lambda_\epsilon}.$$

Then, the distribution of the centered eigenvalue becomes

$$\frac{1}{\epsilon} \left(\frac{1}{\Lambda_\epsilon} - \frac{1}{\Lambda} \right) = \frac{1}{\epsilon} \left(\frac{\Lambda - \Lambda_\epsilon}{\Lambda \Lambda_\epsilon} \right) \quad (4.75)$$

$$\frac{1}{\epsilon} (\Lambda - \Lambda_\epsilon) \sim N(0, \Lambda^4 \Gamma). \quad (4.76)$$

This leads to the following conjecture.

Conjecture 3. *With the current notation,*

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon - \lambda_0}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{\Lambda_0 - \Lambda_\epsilon}{\epsilon} \sim \tilde{G}, \quad (4.77)$$

where \tilde{G} is a mean-zero Gaussian random variable with variance given by

$$\frac{j_a^2 \int_0^1 J_a^2(j_a \sqrt{x}) J_{a+1}^2(j_a \sqrt{x}) dx}{\left(\int_0^1 J_{a+1}^2(j_a \sqrt{x}) dx \right)^2}. \quad (4.78)$$

Since this is the same variance that was found through taking $\beta \rightarrow \infty$ and then $n \rightarrow \infty$, and since Gaussian random variables with the same mean and variance are equal in law, the proof of this conjecture would show that the limits do commute, at least in the case of the smallest eigenvalues.

Remark 14. *From the above, it is plain to see that things are much more complicated in the Laguerre ensembles when compared to the Hermite ensembles. It is natural to ask if the random differential operator approach that lead so quickly to the correct, albeit formal, result for the Hermite ensembles can be applied here. Unfortunately, this seems to be another place where the Laguerre ensembles are much more resistant to analysis. For now, there are many questions which can be answered for the Hermite ensembles which remain conjectures and open problems in the Laguerre ensembles.*

4.5 Transition from Hard Edge to Soft as $a \rightarrow \infty$.

As $a \rightarrow \infty$, Theorem 2.2.3 shows that the smallest eigenvalue of (β, a) -Laguerre transitions from hard edge to soft edge behavior. The following result shows that this is true when the limits are taken in the order $\beta \rightarrow \infty$, then $n \rightarrow \infty$, and finally $a \rightarrow \infty$.

Theorem 4.5.1. *Let $\Lambda_{\infty, a}$ be the smallest eigenvalue of the (β, a) -Laguerre ensemble and λ_∞ the smallest eigenvalue of the β -Hermite ensemble, after $\beta \rightarrow \infty$. Then, as $a \rightarrow \infty$, the Gaussian noise in $\Lambda_{\infty, a}$ converges to the Gaussian noise in λ_∞ .*

The proof is based on the following Lemma:

Lemma 4.5.2. *Let j_a be the first root of $J_a(x)$, the Bessel function of order a , and let a_1 be the first root of $Ai(x)$, the Airy function. Then,*

$$\lim_{a \rightarrow \infty} a^{-4/3} \frac{j_a^2 \int_0^1 J_{a+1}^2(j_a \sqrt{x}) J_a^2(j_a \sqrt{x}) dx}{\left(\int_0^1 J_{a+1}(j_a \sqrt{x}) dx \right)^2} = \frac{\int_0^\infty Ai^4(x + a_1) dx}{\left(\int_0^\infty Ai^2(x + a_1) dx \right)^2} \quad (4.79)$$

Proof of Lemma. Start with the change variables $y = \sqrt{x}$:

$$j_a^2 \int_0^1 J_a^2(j_a \sqrt{x}) J_{a+1}^2(j_a \sqrt{x}) dx = 2j_a^2 \int_0^1 y J_a^2(j_a y) J_{a+1}(j_a y) dy. \quad (4.80)$$

As $a \rightarrow \infty$, the mass of the integrand concentrates around values of y that are close to 1, see figure 4.1.

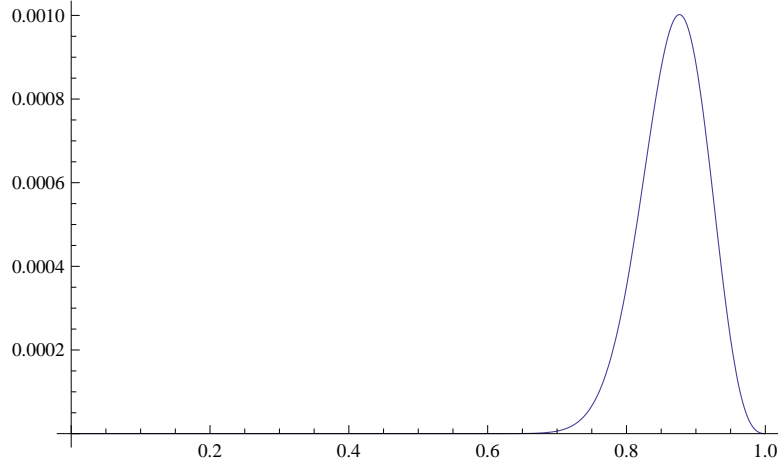


Figure 4.1: Graph of $J_a^2(j_{a,1}\sqrt{x})J_{a+1}^2(j_{a,1}\sqrt{x})$ with $a = 50$.

This suggests breaking the integral into two pieces as follows:

$$2j_a^2 \int_0^1 y J_a^2(j_a y) J_{a+1}(j_a y) dy = 2j_a^2 \int_{1-\frac{\mu}{a^{2/3}}}^1 y J_a^2(j_a y) J_{a+1}^2(j_a y) dy + 2j_a^2 \int_0^{1-\frac{\mu}{a^{2/3}}} y J_a^2(j_a y) J_{a+1}^2(j_a y) dy, \quad (4.81)$$

for an appropriately chosen μ which will later be taken to infinity.

To show that the second integral above is of lower order than the first, use the expansion, valid for $a \rightarrow \infty$ and $\alpha > 0$,

$$J_a(a \operatorname{sech} \alpha) = \frac{e^{a(\tanh \alpha - \alpha)}}{(\frac{1}{2}\pi a \tanh \alpha)^{1/2}} + o(a^{-1}). \quad (4.82)$$

In the regime under consideration, $\operatorname{sech} \alpha < 1$ which means that $0 < \alpha < 1$. For these values of α , $\tanh \alpha - \alpha < 0$, so the exponential term looks like e^{-ca} , for some small positive c . For these values of α ,

$$\lim_{a \rightarrow \infty} \frac{e^{a(\tanh \alpha - \alpha)}}{(\frac{1}{2}\pi a \tanh \alpha)^{1/2}} + o(a^{-1}) = 0, \quad (4.83)$$

which shows that only the first integral on the right-hand side of (4.81) contributes in the limit.

Turn now to the asymptotic behavior of Bessel roots of large order. Define $\rho_a(m) = j_{a,m}$. Then,

$$\rho_a(m) = a + \alpha a^{1/3} + o(a^{-1/3}), \quad (4.84)$$

where α in this context is defined by

$$\frac{Ai(\alpha)}{Bi(\alpha)} = \tan(\pi m). \quad (4.85)$$

Since $\tan(\pi m) = 0$ for $m = 1, 2, \dots$, this makes $\alpha = -a_m/$. Putting this together gives,

$$j_a = a \left(1 - \frac{a_1}{a^{2/3}}\right) + o(a^{-1/3}). \quad (4.86)$$

This indicates that the proper scaling for y is given by $y = 1 - \frac{t}{a^{2/3}}$. Making this change leads to

$$2j_a^2 \int_{1-\frac{\mu}{a^{2/3}}}^1 y J_a^2(j_a y) J_{a+1}^2(j_a y) dy \quad (4.87)$$

$$\begin{aligned} &\sim \frac{2j_a^2}{a^{2/3}} \int_0^\mu J_a^2 \left(a \left(1 - \frac{a_i}{a^{2/3}}\right) \left(1 - \frac{t}{a^{2/3}}\right) \right) J_{a+1}^2 \left(a \left(1 - \frac{a_i}{a^{2/3}}\right) \left(1 - \frac{t}{a^{2/3}}\right) \right) dt \\ &\sim \frac{2j_a^2}{a^{2/3}} \int_0^\mu J_a^2 \left(a \left(1 - \frac{t + a_i}{a^{2/3}}\right) \right) J_{a+1}^2 \left(a \left(1 - \frac{t + a_i}{a^{2/3}}\right) \right) dt \end{aligned} \quad (4.88)$$

The following relationship holds between Bessels of large order and Airy functions:

$$J_a^2(az) \sim \left(\frac{\zeta(z)}{1-z^2} \right)^{1/2} \frac{1}{a^{2/3}} Ai^2 \left(a^{2/3} \zeta(z) \right), \quad (4.89)$$

where $\zeta(t)$ is defined as

$$\zeta^{3/2}(t) = \int_t^1 \frac{\sqrt{1-s^2}}{s} ds. \quad (4.90)$$

In the regime under consideration,

$$\zeta^{3/2}(1-\epsilon) \sim \epsilon^{3/2}, \quad (4.91)$$

so $\zeta(1-\epsilon) \sim \epsilon$. Also, for $z \sim 1$, $1-z^2 \sim 1-z$. Using this, the expansion becomes

$$J_a^2(a(1-\eta(t)/a^{2/3})) \sim \frac{1}{a^{2/3}} Ai^2(a^{2/3}\eta(t)/a^{2/3}), \quad (4.92)$$

where $\eta(t) = t + a_1$. The same holds, as $a \rightarrow \infty$ for the J_{a+1}^2 term as well, so making these changes to the integral gives

$$\int_0^\mu J_a^2 \left(a \left(1 - \frac{t + a_i}{a^{2/3}} \right) \right) J_{a+1}^2 \left(a \left(1 - \frac{t + a_i}{a^{2/3}} \right) \right) dt \sim \frac{1}{a^{4/3}} \int_0^\mu Ai^4(t + a_1) dt. \quad (4.93)$$

Now let $\mu \rightarrow \infty$. Taking the full integral, and recalling how j_a behaves for large a gives

$$\frac{2j_a^2}{a^{2/3}} \int_0^1 y J_{a+1}^2(j_a y) J_a^2(j_a y) dy \Rightarrow 2 \int_0^\infty Ai^4(t + a_1) dt. \quad (4.94)$$

The last thing to do is consider the ratio of the integrals. Since

$$\int_0^1 j J_a^2(j_a y) dy \Rightarrow \frac{1}{a^{2/3}} \int_0^\infty Ai^2(t + a_1) dt, \quad (4.95)$$

and since the scaling terms cancel with the roots in the numerator, there will be an extra $a^{4/3}$ in front of the Airy integrals. This leads to the scaling as given in the theorem.

□

With the proof of Lemma 4.5.2 complete, to prove Theorem 4.5.1, note that it has been established that the variance of the Gaussian random variables converges. Since they are both mean-zero, this means that they have the same distribution and the proof is complete.

Chapter 5

Future Work and Open Problems

5.1 $\beta \rightarrow 0$.

After examining the limiting behavior as $\beta \rightarrow \infty$, it's natural to ask about the degenerate case, that of $\beta \rightarrow 0$. In this situation, the repulsion between the eigenvalues built into the models by the Vandermonde term is disappearing and one would expect that the eigenvalues behavior becomes completely random. Probabilistically, this means that the eigenvalues form a Poisson point process. While this is still an open question, some promising directions for future research are presented.

Both of the possible avenues proposed for exploring this question are based upon the following consideration. Given a solution, $\psi(x, \lambda)$, of the eigenvalue problem

$$\mathfrak{G}_{\beta,a}\psi(x, \lambda) = \lambda\psi(x, \lambda), \tag{5.1}$$

Riccati's map, $p \equiv \psi'/\psi$, transforms the above into a diffusion equation of the form

$$dp(x) = \frac{2}{\sqrt{\beta}}p(x)db(x) + \left(\left(a + \frac{2}{\beta}\right)p(x) - p^2(x) - \lambda e^{-x} \right) dx. \tag{5.2}$$

From Sturm's oscillation theorem, the eigenvalues of $\mathfrak{G}_{\beta,a}$ are counted by the zeros of ψ , which correspond to the places where $p(x)$ hits $-\infty$. Thus, if you can understand the dynamics of the diffusion (5.2), you can understand properties of the eigenvalues. In the present context, this way of looking at the problem leads to the following

$$P(\Lambda_0(\beta, a) > \lambda) = P_{+\infty}(\tau_{-\infty}(p) = \infty). \tag{5.3}$$

Here, $\Lambda_0(\beta, a)$ is the smallest eigenvalue of the (β, a) -Laguerre ensemble, $P_{+\infty}(\tau_{-\infty}(p) = \infty)$ is the probability that the diffusion $p(x)$ started at $+\infty$ never hits $-\infty$. If the conditional distribution of the diffusion $p(x)$ conditioned on it not reaching $-\infty$ was known, then the Cameron-Martin-Girsanov formula, which relates the measure on paths given by two diffusion, could be used to compute $P_{+\infty}(\tau_{-\infty}(p) = \infty)$ explicitly. Unfortunately, this conditional distribution is unknown, but the Cameron-Martin-Girsanov formula can be applied to a new diffusion Y which approximates the given diffusion.

In [VV], Valkó and Virág use the Cameron-Martin-Girsanov transformation to prove a conjecture about large gap probabilities for eigenvalues in the bulk of the spectrum. As it is the technique they used rather than the result that is relevant to the problem under consideration, the statement of the theorem will not be given. Instead, the method used will be described.

Consider two stochastic differential equations

$$dX = g(t, X)dt + dB, \quad \lim_{t \rightarrow 0} X(t) = -\infty \quad (5.4)$$

$$dY = h(t, Y)dt + b\tilde{B}, \quad \lim_{t \rightarrow 0} Y(t) = -\infty \quad (5.5)$$

on the interval $(0, T]$. Let

$$G_s = G_s(X) = \int_0^s h(t, X) - g(t, X)dX - \frac{1}{2} \int_0^s h(t, X)^2 - g(t, X)^2 dt. \quad (5.6)$$

Consider the process \tilde{Y} whose density with respect to the distribution of the process X is given by e^{G_T} . Then \tilde{Y} satisfies the second SDE above and never blows up to $+\infty$ almost surely. Moreover, for any nonnegative function ϕ of the path of X that vanishes when X blows up we have

$$\mathbf{E}\phi(X) = \mathbf{E}[\phi(Y)e^{-G_T(Y)}] \quad (5.7)$$

This means that if you know $G_T(Y)$ and are interested in the probability that X never explodes, you can let $\phi(X)$ be the indicator function for the diffusion not exploding:

$$P(X_t \text{ doesn't explode}) = E(\mathbb{1}_{Y_t \text{ doesn't explode}} e^{-G_T(Y)}) \quad (5.8)$$

Now recall the Ricatti transformation of the diffusion describing the eigenvalues of the general (β, a) -Laguerre ensemble,

$$dp(x) = \frac{2}{\sqrt{\beta}} p(x) db(x) + \left(\left(a + \frac{2}{\beta} \right) p(x) - p^2(x) - \lambda e^{-x} \right) dx. \quad (5.9)$$

Letting $\Lambda_0(a, \beta)$ denote the smallest eigenvalue of this ensemble

$$P(\Lambda_0(a, \beta) > \lambda) = P_\infty(t \rightarrow X_t \text{ never hits } -\infty), \quad (5.10)$$

where

$$dX_t = \frac{2}{\sqrt{\beta}} X_t db_t + \left(\left(a + \frac{2}{\beta} \right) X_t - X_t^2 - \lambda e^{-t} \right) dt \quad (5.11)$$

Transform the SDE given by X_t , by first setting $c = \beta/4$ and $Y_t = X_{ct}$. Next, let $Z_t = \log Y_t$.

Finally, let $W_t = Z_t + (\beta/8)t - 1/2 \log \lambda$. Then

$$dW_t = dZ_t + \frac{\beta}{8} dt = db_t + \left(\frac{\beta}{4} \left(a + \frac{1}{2} \right) - \frac{\beta}{4} \sqrt{\lambda} e^{-\frac{\beta}{8}t} \cosh(W_t) \right) dt \quad (5.12)$$

From this it follows that

$$P(X_t \text{ doesn't explode to } -\infty) = \mathbf{E}(e^{-G_t(W)}) \quad (5.13)$$

With this background, consider the following SDE:

$$dX_t = db_t + \left(\frac{\beta}{4} \left(a + \frac{1}{2} \right) - \frac{\beta}{4} \sqrt{\lambda} e^{-\frac{\beta}{8}t} \cosh(X_t) \right) dt \quad (5.14)$$

If one were able to find an appropriate function $h(\beta, \lambda, a)$ such that the SDE

$$dY_t = \tilde{b}_t + h(\beta, \lambda, a) dt, \quad (5.15)$$

fit into the framework of the Girsanov-Cameron-Martin theory developed above, then the hope is that the technique used in [VV] could be applied. Unfortunately, all efforts to find $h(\beta, \lambda, a)$ have been unsuccessful.

Another possible approach is based on the following result of H.P. McKean.

Theorem 5.1.1. ([McK94]) *Let $\Lambda(L)$ be the smallest eigenvalue of the operator $D^2 - b'$, with D differentiation and b' white noise, on a circle $0 \leq x \leq L$ of large perimeter L . Then,*

$$\lim_{L \uparrow \infty} P[(L/\pi)\Lambda^{1/2}(L) \exp(-\frac{8}{3}\Lambda^{3/2}(L)) > x] = e^{-x} \text{ for } x \geq 0. \quad (5.16)$$

This proves that, as $L \rightarrow \infty$, the appropriately scaled smallest eigenvalue becomes an exponentially distributed random variable, $\sim \text{Exp}(1)$.

The proof of this theorem is based upon a Riccati transformation and diffusion representation, much like the results in [RR09]. The connection to the $\beta \rightarrow 0$ case is through the identification of L with $2/\sqrt{\beta}$. Then, as $L \rightarrow \infty$, $\beta \rightarrow 0$. This leads to the following conjecture:

Conjecture 4. *Let $\Lambda(\beta)$ denote the smallest eigenvalue of the (β, a) -Laguerre ensemble. Then,*

$$\lim_{\beta \rightarrow 0} P[\frac{2}{\pi\beta}\Lambda^{1/2}(\beta) \exp(-\frac{8}{3}\Lambda^{3/2}(\beta)) > x] = e^{-x} \text{ for } x \geq 0. \quad (5.17)$$

If this conjecture is true, then the smallest eigenvalue of the (β, a) -Laguerre ensemble behaves like an exponentially distributed random variable as $\beta \rightarrow 0$. If one were able to prove that the k smallest eigenvalues, for any k , were distributed as independent exponential random variables, then it would be established that the eigenvalues form a Poisson point process in the limit $\beta \rightarrow 0$.

5.2 Resolvent

In [RR09], the integral operator (2.47) was found by considering the inverse of the discretized matrix operator $M_{\beta,a}$, and then passing to the continuum limit. This corresponds to evaluating the resolvent $R(z) = (M_{\beta,a}M_{\beta,a}^T - z\text{Id})^{-1}$ at $z = 0$. In order to motivate the work that follows in the attempt to extend this result to a limiting resolvent for all z , the general steps in the derivation of $\mathfrak{G}_{\beta,a}^{-1}$ will be outlined here.

Let

$$M_{\beta,a} = \frac{1}{\sqrt{\beta}} \begin{pmatrix} \chi_{(a+1)\beta} & & & & \\ -\chi_{\beta} & \chi_{(a+2)\beta} & & & \\ & -\chi_{2\beta} & \chi_{(a+3)\beta} & & \\ & & \ddots & \ddots & \\ & & & -\chi_{(n-1)\beta} & \chi_{(a+n)\beta} \end{pmatrix} \quad (5.18)$$

where all the chi random variables along the diagonal are independent from those along the subdiagonal.

If $B = (b_{i,j})$ is the lower bidiagonal matrix

$$B = \begin{pmatrix} b_{1,1} & & & \\ b_{2,1} & b_{2,2} & & \\ & b_{3,2} & b_{3,3} & \\ & & \ddots & \ddots \end{pmatrix}, \quad (5.19)$$

the entries of the inverse are given by

$$\tilde{b}_{i,j} = [B^{-1}]_{i,j} = \frac{(-1)^{i+j}}{b_{i,i}} \prod_{k=j}^{i-1} \frac{b_{k+1,k}}{b_{k,k}} \quad \text{for } j < i. \quad (5.20)$$

The next thing to consider is the following operator which embeds any $n \times n$ matrix $A = (a_{i,j})$ into $L^2[0, 1]$ without changing the spectrum:

$$(Af)(x) = \sum_{j=1}^n a_{i,j} n \int_{x_{j-1}}^{x_j} f(y) dy \quad \text{for } x_{i-1} \leq x \leq x_i \quad (5.21)$$

with $x_i = i/n$.

Consider the action of this operator on the matrix $(nM_{\beta,a}M_{\beta,a}^T)^{-1} = (n^{1/2}M_{\beta,a}^T)^{-1}(n^{1/2}M_{\beta,a})^{-1}$.

The second term in the product after the embedding (5.21) becomes

$$\left((n^{1/2}M_{\beta,a})^{-1} f \right) (x) = \sum_{j=1}^{\lfloor nx \rfloor} \frac{\sqrt{\beta n}}{\chi_{(\lfloor nx \rfloor + a)\beta}} \prod_{k=j}^{\lfloor nx \rfloor - 1} \frac{\chi_{k\beta}}{\chi_{(k+a)\beta}} \int_{x_{j-1}}^{x_j} f(y) dy. \quad (5.22)$$

This shows that $n^{-1/2}M_{\beta,a}^{-1}$ can be thought of as an integral operator $K_{\beta,a}^n$ with discrete kernel

$$k_{\beta,a}^n(x, y) = \frac{\sqrt{\beta n}}{\chi_{(i+a)\beta}} \exp \left\{ \sum_{k=j}^{i-1} \log \chi_{k\beta} - \log \chi_{(k+a)\beta} \right\} \mathbb{1}_L(x, y) \quad (5.23)$$

where $\mathbb{1}_L = \mathbb{1}_{\{x_{i-1} \leq x < x_i\}} \mathbb{1}_{\{x_{j-1} \leq y < x_j\}}$ and $i > j$.

The convergence result that is needed to pass to the continuum limit is given by the following lemma.

Lemma 5.2.1. (Lemma 5 of [RR09]) *There is a Brownian motion $b(\cdot)$ such that for $y < x$ lying in $(0, 1]$*

$$\frac{\sqrt{\beta n}}{\chi(\lfloor nx \rfloor + a)\beta} \Rightarrow \frac{1}{\sqrt{x}} \quad (5.24)$$

and

$$\sum_{k=\lfloor ny \rfloor}^{\lfloor nx \rfloor} \log \chi_{k\beta} - \log \chi_{(k+a)\beta} \Rightarrow (a/2) \log(y/x) + \int_y^x \frac{db_z}{\sqrt{\beta z}}, \quad (5.25)$$

in law in the Skorohod topology.

From this result, the $n \rightarrow \infty$ continuum limit of the discrete kernel $k_{\beta,a}^n$ is seen to be

$$k_{\beta,a}(x, y) \equiv x^{-\frac{1+a}{2}} \exp \left[\int_y^x \frac{db_z}{\sqrt{\beta z}} \right] y^{a/2} \mathbb{1}_{y < x}. \quad (5.26)$$

Carrying out the same program for $n^{-1/2} M_{\beta,a}$ and writing the eigenvalue problem as $f(x) = \lambda \left((K_{\beta,a}^T K_{\beta,a}) f \right) (x)$ (recall that this integral operator is the inverse of the differential operator for which the eigenvalue problem is initially defined), the explicit form then reads

$$f(x) = \lambda \int_x^1 x^{a/2} e^{\int_x^y \frac{db_s}{\sqrt{\beta s}}} y^{-(a+1)} \int_0^y e^{\int_z^y \frac{db_s}{\sqrt{\beta s}}} z^{a/2} f(z) dz dy \quad (5.27)$$

$$= \lambda \int_0^1 (xy)^{a/2} \left(\int_{x \vee y}^1 e^{-2 \int_z^1 \frac{db_s}{\sqrt{\beta s}}} z^{-(a+1)} dz \right) e^{\int_x^1 \frac{db_s}{\sqrt{\beta s}}} e^{\int_y^1 \frac{db_s}{\sqrt{\beta s}}} f(y) dy \quad (5.28)$$

after an integration by parts. Now, make the substitution $g(x) = x^{-a/2} e^{-\int_x^1 \frac{db_s}{\sqrt{\beta s}}} f(x)$, the time-change $\int_x^1 s^{-1/2} db_s = \hat{b}(\log(1/x))$, and the change of variables $(x, y) \rightarrow (e^{-x}, e^{-y})$. Carrying out these changes finishes the derivation of $\mathfrak{G}_{\beta,a}^{-1}$.

When $z \neq 0$, the tridiagonal matrix $M_{\beta,a} M_{\beta,a}^T - z \text{Id}$ cannot be factored into the product of two bidiagonal matrices. This means that the tridiagonal matrix itself must be inverted, which is

much more complicated than in the bidiagonal case. In what follows, the above derivation will be considered from this point of view. Since the $z = 0$ case is easier, it will be considered first. The general $z \neq 0$ situation will be addressed.

Return to the formula for the inverse of a bidiagonal matrix, (5.20). From this it follows that

$$(BB^T)_{i,j}^{-1} = \sum_{k=i \vee j}^n \tilde{b}_{k,i} \tilde{b}_{k,j}. \quad (5.29)$$

Consider the following sequences connected to B and B^T :

$$p_1(0) = 1, \quad p_2(0) = \frac{b_{1,1}}{b_{2,1}}, \quad \text{and} \quad p_i(0) = \prod_{l=1}^{i-1} \frac{b_{l,l}}{b_{l+1,l}} \quad (5.30)$$

$$q_1(0) = \frac{1}{b_{1,1}}, \quad q_2(0) = \frac{b_{2,1}}{b_{2,2}b_{1,1}}, \quad \text{and} \quad q_i(0) = \frac{1}{b_{i,i}} \prod_{l=1}^{i-1} \frac{b_{l+1,l}}{b_{l,l}}. \quad (5.31)$$

It can be checked that

$$(B^{-1})_{i,j} = (-1)^{i+j} p_i(0) q_j(0), \quad i \geq j, \quad (\text{lower triangular}) \quad (5.32)$$

$$(B^T)^{-1}_{i,j} = (-1)^{i+j} p_i(0) q_j(0), \quad j \geq i, \quad (\text{upper triangular}). \quad (5.33)$$

From this, the following relationship exists:

$$(BB^T)_{i,j}^{-1} = \sum_{k=i \vee j}^n \tilde{b}_{k,i} \tilde{b}_{k,j}. \quad (5.34)$$

$$= p_i(0) p_j(0) \sum_{k=i \vee j}^n q_k^2(0). \quad (5.35)$$

Let $B = M_{\beta,a}$, and embed the matrix $(nBB^T)^{-1}$ into $L^2[0,1]$ using (5.21). This gives

$$((nBB^T)^{-1}f)(x) = \sum_{k=1}^n \beta (BB^T)^{-1}_{i,k} f(y_k) \Delta_k, \quad (5.36)$$

where $(i-1)/n \leq x < i/n$, $\Delta_k = 1/n$, and $f(y_k)$ is the discretization of f . Now

$$\begin{aligned} (BB^T)^{-1}_{i,k} &= ((B^T)^{-1}B^{-1})_{i,k} = \sum_{j=i \vee k}^n \tilde{b}_{j,i} \tilde{b}_{j,k} \\ &= \sum_{j=i \vee k}^n \left(\frac{(-1)^{i+j}}{b_{j,j}} \prod_{l=i}^{j-1} \frac{b_{l+1,l}}{b_{l,l}} \right) \left(\frac{(-1)^{k+j}}{b_{j,j}} \prod_{l=k}^{j-1} \frac{b_{l+1,l}}{b_{l,l}} \right) \end{aligned} \quad (5.37)$$

where $\tilde{b}_{i,j}$ is the (i,j) -th entry of B^{-1} which is given in terms of the entries $b_{i,j}$ of B . This gives

$$\begin{aligned}
((nBB^T)^{-1}f)(x) &= \sum_{k=1}^n \beta(BB^T)_{i,k}^{-1} f(z_k) \Delta_k \\
&= \sum_{k=1}^n \beta \sum_{j=i \vee k}^n \left(\frac{(-1)^{i+j}}{b_{j,j}} \prod_{l=i}^{j-1} \frac{b_{l+1,l}}{b_{l,l}} \right) \left(\frac{(-1)^{k+j}}{b_{j,j}} \prod_{l=k}^{j-1} \frac{b_{l+1,l}}{b_{l,l}} \right) f(y_k) \Delta_k \\
&= \sum_{k=1}^n \left(\sum_{j=i \vee k}^n \left(\frac{\sqrt{\beta n}}{b_{j,j}} \prod_{l=i}^{j-1} \frac{b_{l+1,l}}{b_{l,l}} \right) \left(\frac{\sqrt{\beta n}}{b_{j,j}} \prod_{l=k}^{j-1} \frac{b_{l+1,l}}{b_{l,l}} \right) \Delta_j \right) f(y_k) \Delta_k
\end{aligned}$$

Remembering that $i = \lfloor nx \rfloor$, $k = \lfloor ny \rfloor$, and $j = \lfloor nz \rfloor$ and what the entries of the original bidiagonal matrix B are, the inner sum becomes

$$\begin{aligned}
&\sum_{j=i \vee k}^n \left(\frac{\sqrt{\beta n}}{b_{j,j}} \prod_{l=i}^{j-1} \frac{b_{l+1,l}}{b_{l,l}} \right) \left(\frac{\sqrt{\beta n}}{b_{j,j}} \prod_{l=k}^{j-1} \frac{b_{l+1,l}}{b_{l,l}} \right) \Delta_j \Rightarrow \\
&(xy)^{a/2} \exp \left(\int_x^1 \frac{db_s}{\sqrt{\beta s}} \right) \exp \left(\int_y^1 \frac{db_s}{\sqrt{\beta s}} \right) \int_{x \vee y}^1 z^{-(a+1)} \exp \left(-2 \int_z^1 \frac{db_s}{\sqrt{\beta s}} \right) dz \quad (5.38)
\end{aligned}$$

which is exactly the integrand in (5.28). It's interesting to note that this expression was originally derived after doing an integration by parts, but that step is bypassed here.

For the $z \neq 0$ case, some theory on the inversion of symmetric tridiagonal matrices must first be presented. Start with an infinite symmetric tridiagonal matrix H :

$$H = \begin{pmatrix} a_0 & b_0 & & \\ b_0 & a_1 & b_1 & \\ & b_1 & a_2 & \\ & & \ddots & \ddots \end{pmatrix} \quad (5.39)$$

The a_n are assumed to be real and the b_n to be real and positive.

To invert this matrix, construct two sets of orthogonal polynomials $\{P_n(z)\}$ and $\{Q_n(z)\}$ that satisfy the following three-term recurrence

$$zs_n(z) = b_{n-1}s_{n-1}(z) + a_ns_n(z) + b_ns_{n+1}(z), \quad (5.40)$$

with the following initial conditions:

$$P_0(z) = 1, \quad P_1(z) = \frac{z - a_0}{b_0} \quad (5.41)$$

n	$P_n(z)$	$Q_n(z)$
0	1	0
1	$\frac{z-a_0}{b_0}$	$\frac{1}{b_0}$
2	$\frac{1}{b_0 b_1}((z-a_1)(z-a_0) - b_0^2)$	$\frac{1}{b_0 b_1}(z-a_1)$
3	$\frac{1}{b_0 b_1 b_2}((z-a_0)(z-a_1)(z-a_2) - b_0^2(z-a_2) - b_1^2(z-a_1))$	$\frac{1}{b_0 b_1 b_2}((z-a_1)(z-a_2) - b_1^2)$

Table 5.1: Polynomials generated by tridiagonal recurrence

$$Q_0(z) = 0, \quad Q_1 = \frac{1}{b_0}. \quad (5.42)$$

In the language of orthogonal polynomials, the $P_n(z)$'s and $Q_n(z)$'s are the orthogonal polynomials of the first and second kind, respectively. The first few of these polynomials are given in Table 5.1.

Define the resolvent, or Green's function, $G(z)$ through the identity

$$(H - z\text{Id})G(z) = \text{Id}, \quad (5.43)$$

from which it follows that $G(z) = (H - z\text{Id})^{-1}$. The entries of $G(z)$ are given by (see [YA97])

$$G_{nm}(z) = P_n(z)P_m(z) \left(G_{00}(z) + \frac{Q_{n \vee m}(z)}{P_{n \vee m}(z)} \right), \quad (5.44)$$

where $x \vee y$ is the maximum of x and y , and

$$G_{00}(z) = - \lim_{n \rightarrow \infty} \left(\frac{Q_n(z)}{P_n(z)} \right). \quad (5.45)$$

By truncating the infinite matrix, the resolvent for a finite symmetric tridiagonal matrix can be obtained. Starting now with the $N \times N$ matrix H_N with $\{a_n\}_{n=0}^{N-1}$ along the main diagonal and $\{b_n\}_{n=0}^{N-2}$ along the off-diagonals, the resolvent $G_N(z)$, which is symmetric, has entries

$$(G_N(z))_{nm} = P_n(z)P_m(z) \left(\frac{Q_m(z)}{P_m(z)} - \frac{Q_N(z)}{P_N(z)} \right) \quad \text{with } 0 \leq n \leq m \leq N-1. \quad (5.46)$$

Notice that this decouples into two matrices $G_N^{(1)}(z)$ and $G_N^{(2)}(z)$, with

$$G_N(z) = G_N^{(1)}(z) + G_N^{(2)}(z) \quad (5.47)$$

$$(G_N^{(1)}(z))_{nm} = Q_m(z)P_n(z) \quad (5.48)$$

$$(G_N^{(2)}(z))_{nm} = -\frac{Q_N(z)}{P_N(z)}P_n(z)P_m(z). \quad (5.49)$$

Written as matrices, this gives

$$G_N(z) = \begin{pmatrix} Q_0(z)P_0(z) & Q_1(z)P_0(z) & \cdots \\ Q_1(z)P_0(z) & Q_1(z)P_1(z) & \cdots \\ Q_2(z)P_0(z) & Q_2(z)P_1(z) & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix} - \frac{Q_N(z)}{P_N(z)} \begin{pmatrix} P_0^2(z) & P_0(z)P_1(z) & \cdots \\ P_0(z)P_1(z) & P_1^2(z) & \cdots \\ P_0(z)P_2(z) & P_1(z)P_2(z) & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix}$$

Using this and the embedding (5.21) gives

$$\left(\frac{1}{n} (M_{\beta,a} M_{\beta,a}^T - z \text{Id})^{-1} f \right) (x) = \sum_{k=0}^{N-1} \beta (M_{\beta,a} M_{\beta,a}^T - z \text{Id})_{i,k}^{-1} f(y_k) \Delta_k \quad (5.50)$$

$$= \sum_{k=0}^{N-1} \beta (G_N)_{i,k} f(y_k) \Delta_k. \quad (5.51)$$

This can be rewritten as

$$\beta \left(Q_i(z) \sum_{k=0}^i P_i(z) f(y_k) + P_i(z) \sum_{k=i+1}^{N-1} Q_k(z) f(y_k) - \frac{P_i(z) Q_N(z)}{P_N(z)} \sum_{k=0}^{N-1} P_i(z) f(y_k) \right) \Delta_k \quad (5.52)$$

If there is enough structure in the sums of these orthogonal polynomials, the hope is that this will converge to an integral operator in a similar manner to the $z = 0$ case. The next result shows that there is some nice structure to these sums.

Lemma 5.2.2. *Let $P_i(z)$ and $Q_i(z)$ be the orthogonal polynomials of first and second kind associated with the matrix (5.39). Then $\sum_{i=0}^n P_i(z) = \det \tilde{P}_n$, where*

$$\tilde{P}_n = \begin{pmatrix} z - a_0 & -1 & 0 & \cdots & 0 \\ -b_0^2 & z - a_1 & -1 & 0 & \cdots \\ 0 & -b_1^2 & z - a_2 & -1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & -b_{n-1}^2 & z - a_{n-1} & -1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \quad (5.53)$$

and $\sum_{i=0}^n Q_i(z) = \det \tilde{Q}_n$, where

$$\tilde{Q}_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & z - a_1 & -1 & 0 & \cdots \\ 0 & -b_1^2 & z - a_2 & -1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & -b_{n-1}^2 & z - a_{n-1} & -1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}. \quad (5.54)$$

Proof. The proof is based on a recursion given in [Smi65]. Let the orthogonal polynomials $p_i(z)$ be given by the following three-term recurrence:

$$p_i(z) = (\gamma_i z - \alpha_i) p_{i-1}(z) - \beta_i p_{i-2}(z) \quad \text{for } i \geq 2,$$

and

$$p_0(z) = \gamma_0, \quad p_1(z) = (\gamma_1 z - \alpha_1) p_0(z).$$

Define $\{B_i\}$ as follows:

$$B_i = 0 \quad \text{for } i > n,$$

$$B_i = c_i + (\gamma_{i+1} z - \alpha_{i+1}) B_{i+1} - \beta_{i+2} B_{i+2} \quad \text{for } 0 \leq i \leq n.$$

Then

$$f(z) := \sum_{i=0}^n c_i p_i(z) = \gamma_0 B_0.$$

Changing notation to match with the notation used above and performing induction on the Laplace expansion for determinants leads to the result.

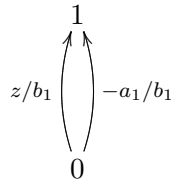
□

Finally, it should be noted that these polynomials can also be realized as the sum of paths along certain weighted graphs. This leads to a graph-theoretic representation. It would not be hard to believe that the Gessel-Viennot Lemma [Aig07, p. 217] is lurking somewhere just below the surface.

It is probably easiest to understand the graph-theoretical representation by giving a few examples. For $P_n(z)$, draw a vertical graph with $n+1$ nodes labelled $0, 1, \dots, n$ increasing vertically. When the node labelled 2 is reached, start a new column, with $n-1$ nodes, labelled $0, 1, \dots, n-2$ increasing vertically. Treat this column exactly the same as the first column. Repeat this process until no more columns can be added. Each node is connected to the one above it by 2 edges, while each node is connected to the node to the left of it by 1 node. All nodes labelled 0 are starting points. The node labelled n is the ending point. Vertical paths can be travelled in two possible ways; horizontal paths in only one. As you move up paths, the terms are multiplied; as you move across paths, the terms are added. The polynomial $P_n(z)$ is found by summing all possible paths from all nodes labelled 0 to the node labelled n . The number of paths from all nodes labelled 0 and the node labelled n , corresponding to the number of terms in the sum for $P_n(z)$, grows as A_{n-1} , where A_n are the Pell numbers, given by the recursion $A_0 = 0, A_1 = 1, A_n = 2A_{n-1} + A_{n-2}$. Examples follow.

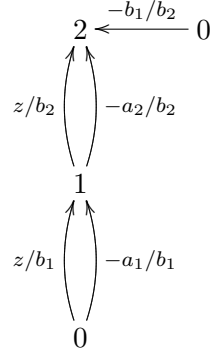
To compute $P_1(z)$, refer to the following graph and sum the paths from 0 to 1. This gives

$$P_1(z) = \frac{z}{b_1} + \frac{-a_1}{b_1} = \frac{z - a_1}{b_1}.$$



For $P_2(z)$, use the following graph. Again, sum along all possible paths from 0 to 2. This gives

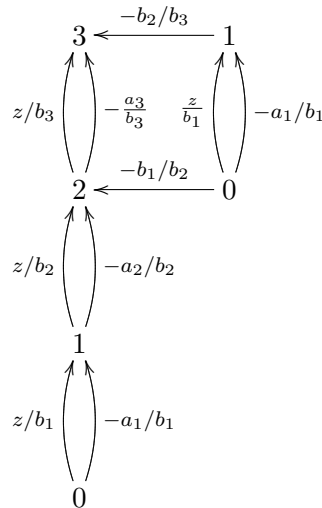
$$P_2(z) = \frac{z^2}{b_1 b_2} + \frac{-a_1 z}{b_1 b_2} + \frac{-a_2 z}{b_1 b_2} + \frac{a_1 a_2}{b_1 b_2} + \frac{-b_1}{b_2} = \frac{1}{b_1 b_2} ((z - a_1)(z - a_2) - b_1^2)$$



For $P_3(z)$, use the following figure. Notice that this illustrates the addition of a new column.

Summing along the paths according to the prescribed rules gives

$$\begin{aligned}
 P_3(z) &= \frac{z^3}{b_1 b_2 b_3} + \frac{-a_1 z^2}{b_1 b_2 b_3} + \frac{-a_2 z^2}{b_1 b_2 b_3} + \frac{-a_3 z^2}{b_1 b_2 b_3} \\
 &+ \frac{a_1 a_2 z}{b_1 b_2 b_3} + \frac{a_1 a_3 z}{b_1 b_2 b_3} + \frac{a_2 a_3 z}{b_1 b_2 b_3} + \frac{-b_1 z}{b_2 b_3} + \frac{-b_2 z}{b_1 b_3} + \frac{-a_1 a_2 a_3}{b_1 b_2 b_3} + \frac{a_1 b_2}{b_1 b_3} + \frac{a_3 b_1}{b_2 b_3} \\
 &= \frac{1}{b_1 b_2 b_3} (z^3 + (-a_1 - a_2 - a_3)z^2 + (a_1 a_2 + a_1 a_3 + a_2 a_3 - b_1^2 b_2^2)z - a_1 a_2 a_3 + a_1 b_2^2 + a_3 b_1^2) \\
 &= \frac{1}{b_1 b_2 b_3} ((z - a_1)(z - a_2)(z - a_3) - b_2^2(z - a_1) - b_1^2(z - a_3))
 \end{aligned}$$



As can be seen from referring to Table 5.1, this produces the correct $P_n(z)$. This reveals more of the combinatorial and graph-theoretical structure of these orthogonal polynomials. The

hope is that this structure can be utilized in the analysis of the resolvent, (5.52). So far this has not been realized, although these methods have reproduced the $z = 0$ case in perhaps a more straight-forward manner.

Bibliography

- [Aig07] Martin Aigner. *A Course in Enumeration*. Springer-Verlag, 2007.
- [BO99] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer, 1999.
- [DE02] Ioana Dumitriu and Alan Edelman. Matrix models for beta-ensembles. *J. Math. Phys.*, 43:5830–5847, 2002.
- [DE05] Ioana Dumitriu and Alan Edelman. Eigenvalues of Hermite and Laguerre ensembles: Large beta asymptotics. *Ann. I. H. Poincaré*, 41:1083–1099, 2005.
- [Dem97] James W. Demmel. *Applied Numerical Linear Algebra*. Society for Industrial and Applied Mathematics, 1997.
- [EK86] S. Either and T. Kurtz. *Markov Processes*. Wiley Series in Probability and Statistics, 1986.
- [ES07] Alan Edelman and Brian Sutton. From random matrices to stochastic operators. *J. Stat. Phys.*, 127:1121–1165, 2007.
- [GR94] I.S. Gradshteyn and I.M. Ryzhik. *Tables of Integrals, Series, and Products*. Academic Press, 1994.
- [McK94] H.P. McKean. A limit law for the ground state of Hill’s equation. *Jour. of Stat. Phys.*, 74:1227– 1232, 1994.
- [MP67] V.A. Marchenko and L.A. Pastur. Distribution of eigenvalues for some sets of random matrices. *Mat. Sb. (N.S.)*, 72:507–536, 1967.
- [OLBC10] Frank W.J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.
- [RR09] José Ramírez and Brian Rider. Diffusion at the random matrix hard edge. *Comm. Math. Phys.*, 288:887–906, 2009.
- [RRV] José Ramírez, Brian Rider, and Bálint Virág. Beta ensembles, stochastic Airy spectrum, and a diffusion. *To appear in Jour. Amer. Math. Soc.* Also available at *arXiv:math/0607331*.

- [Smi65] Francis J. Smith. An algorithm for summing orthogonal polynomial series and their derivatives with applications to curve-fitting and interpolation. *Mathematics of Computation*, 19:33–36, 1965.
- [Sze03] Gabor Szegő. *Orthogonal Polynomials*. American Mathematical Society, 2003.
- [Tro84] H.F. Trotter. Eigenvalue distributions of large hermitian matrices; Wigner’s semicircle law and a theorem of Kac, Murdock, and Szegő. *Adv. in Math.*, 54:67–82, 1984.
- [TW94a] Craig Tracy and Harold Widom. Level spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159:151–174, 1994.
- [TW94b] Craig Tracy and Harold Widom. Level spacing distributions and the Bessel kernel. *Comm. Math. Phys.*, 161:289–309, 1994.
- [VV] Benedek Valkó and Bálint Virág. Large gaps between random eigenvalues. *To appear in Annals of Probability*. Also available at *arXiv:0811.0007v3*.
- [Wig55] Eugene Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Ann. of Math.*, 62:548–564, 1955.
- [YA97] H.A. Yamani and M.S. Abdelmonem. The analytic inversion of any finite symmetric tridiagonal matrix. *J. Phys. A: Math. Gen.*, 30:2889–2893, 1997.